AN EFFICIENT OPERATOR SPLITTING TECHNIQUE FOR OPTION PRICING UNDER STOCHASTIC VOLATILITY

MSc. (MATHEMATICAL SCIENCES) THESIS

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UNIVERSITY OF MALAWI

CHANCELLOR COLLEGE

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MSc. (MATHEMATICAL SCIENCES) THESIS

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DECLARATION

I, the undersigned, hereby declare that this thesis is my own original work which has not been submitted to any other institution for similar purposes. Where other people's work has been used, acknowledgements have been made.

<u>**PATRICK CHIDZALO**</u>

Full Legal Name

Signature

Date

CERTIFICATE OF APPROVAL

The undersigned certify that this thesis represents the student's own work and effort and has been submitted with our approval.

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ABSTRACT

Stochastic Volatility models are more realistic than the Black-Scholes model for Vanilla option pricing, because risk neutrality assumption is avoided. We solve Heston's stochastic volatility model, that has been solved using Monte-Carlo method which is computationally realistic in the third dimension only; and Finite difference methods that are more efficient, but face curse of dimensionality in at least three dimensions. This study adopts the Strang's operator splitting method that uses stochastically weighted sub-problems. Fourth order Runge-Kutta method for partial differential equations are used within finite difference formulations of the split sub-problems. Charpit method is applied in the nonlinear part of the Option pricing model. The method has second order truncation error and is computationally more realistic, which is verified by the Chicago Board of Options Exchange data. The running time of computer powered simulations are lower than those found in the two methods.

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ABBREVIATIONS AND ACRONYMS

1-D	One Dimension
2-D	Two Dimensions
3-D	Three Dimensions
CBOE	Chicago Board of Option Exchange
CCF	Cross Correlation Function
FDM	Finite Difference Method
KdV	Kortewed-de Vries equation
MC	Monte Carlo Method
$\mathcal{O}(f(h))$	Trunction error under Taylor series approximation operated by $f(h)$
OSM	Operator Splitting Method
SV	Stochastic Volatility Model
SAA	South African Airways
YF	Yahoo Finance

Chapter 1

INTRODUCTION

1.1 Background

1.1.1 Options and hedging

Any investor has the preference to divide his or her wealth into investing in a number of assets. A combination of such assets with an expectation of returns is called portfolio (Joshi, 2008). It is very imperative to invest in both risk and non risky assets-a classification based on the likelihood of quantifiable returns (Ross, 2009). If the event of obtaining a positive return is certain, the investment is said to be risk free. A premium bond and a bank fixed deposit are good examples of risk free assets.

A sensible motivation for adopting a risky investment is always when its internal rate of positive return is greater than that realized in a risk-free asset (Chertock, 2010). The rate of returns defining the value of an asset is dictated by nature of the market. One market driver is politics, which in turn influence speculations. For example, the firing the finance minister of the Republic of South Africa, in 2016, lead to the rand's plummeting by a degree of 5% with respect to United

States Dollar (Hozman & Tichy, 2016) . The speculation that financial environment would be unfavorable owing to change brought by new appointments lead to an increased withdrawal of United States Dollar, increasing the supply of the Rand which lowered the Rand's value.

Within the portfolios, different investment schemes are used to offset the risk of adverse movement of prices of assets. For example, a transport company that uses diesel bought from a diesel company may adopt an investment scheme to avoid risks of adverse change in the price of the fuel. An investment scheme that is used to offset the effect of adverse movement of a price is called a hedge, and investing in such a scheme is called hedging (Ross, 2009). A bad hedge leads to losses of returns. A good hedge leads to gains in returns. These hedging schemes usually employ a combination of buying and selling of assets, that are called financial derivatives, whose prices are affected by prices of other assets known as the underlying (Gomes & Michaelides, 2007).

The most common financial derivatives are options, forwards and futures. A forward contract is a customized agreement to buy or sell an asset at a price fixed today (Joshi , 2008). If all parties in a market agree on common guidelines of buying and selling of forward contracts, in other words "if forward contracts are standardized", then those forward contracts are called futures contracts or simply futures (Langtangen, 2012). Unlike the forward contracts, options give the holder the rights to buy or sell a fixed quantity of an asset at a price fixed today. The right to buy is called a call option and the right sell is a put option (Hull, 2000).

Using an option or a contract is called exercising it. The price which is guaranteed by the option is called strike price (An et al., 2008). To exercise an option there are many rules and the simplest of them is called the European option which can be exercised on one specific date in future. An American option can be exercised any day before the exercise date (Ross, 2009). If no further assumptions are made to these two rules, then they are called Vanilla or Ordinary options.

The holder of an option is not obliged to exercise it on the exercise date. Hence, if the underlying price rises or falls, the holder of an option may decide whether to exercise the option or not. Thus, an option can give the holder more freedom to exercise or not than other forms of derivatives (Schweizer, 2010).

There are a lot tangible evidence to this fact. For example, in the years around 2000, the price of jet fuel was increasing, and therefore it was predicted and speculated that the prices would increase in the next decade. Owing to this, the South African Airways (SAA) went into a futures contract with ICE Brent Oil Company, that provided for it to be buying a fixed amount of fuel every year for more than half a decade (SAA, 2014). In the years starting from 2004, the global trends in the fuel prices started to lower greatly than the agreed strike price.

Since the SAA had gone into a futures, it kept buying the fuel at an expensive price while its competing companies took the advantage of the lowering prices of fuel and hence made more profits. The SAA could not pay dividends to its shareholders this time and it is reported that the company made losses amounting to 8 billion Rands due to the bad fuel hedge (Thomas, 2015). Thus, the company would have adopted an option as opposed to a futures contract.

In Malawi, pre-qualification of suppliers, (advertised in almost every news paper), is a common means in which options are traded. The holder of the contract is always benefiting in that goods are supplied to the holder at constant price throughout the year. The supplier can be at a loss if prices rise higher than the agreed price.

Since options can as well be used for hedging, they have been the most traded derivatives since the earliest times. From the years around 1973 up to the moment, option trade has grown up to the liquidity (that is trade volume) of at least 6 billion US dollars (Shah et al., 2012). Thus, option trade has a greater influence to the derivatives market.

1.1.2 Ways of Option Pricing

However, by increasing liquidity in the market, the options have lead to an increase in transaction costs. This phenomenon is called bid-offer spread. In other words, options have increased spreads in the stock markets (Joshi , 2008). This situation has been aggravated by the fact that most investors fail to understand options, and hence spreads may lead to a potential loss in returns, especially if options are not priced properly (Ross, 2009). In addition, being flexible means they should be priced correctly and fairly to see them being honoured by the holders (Fodya, 2007). Different models and theories have been developed to price options fairly in order to see them honoured on the exercise date. The most common are the Binomial, the Black-Scholes, and the Stochastic Volatility models (Fodya, 2007).

Most of these models and theories have been constructed under the assumptions that a market is not moving , that is, actions of buyers and sellers cannot change the price of a market; secondly, that there is existence of liquidity in the market, that is, stock can be bought or sold at any time whenever one wishes; thirdly, that an investor can go short, that is, to sell a stock that one does not own, the opposite of which is to go long; and lastly, that there are no transaction costs, the opposite of which is the bid-offer spread (Ross, 2009). The last assumption leads to a sure-win betting scheme which is called arbitrage (Ketcheson, 2011). The rest assumptions are model restricted (Joshi , 2008). Clearly, most of these assumptions are far from being realistic.

The Black-Scholes and Binomial models are the primary models that are commonly used to price options using available software in many financial institutions (Joshi , 2008). Both of these models are based on the same theoretical assumptions that all risks can be eliminated, in other words "there is risk-neutral evaluation ", and that given that the initial price of underlying price is known or specified, the next change in price is dependent on the present change but independent of the past (Kuo, 2002). The second assumption is known as Brownian motion , and the memory-less property in the second assumption is called the Markov property (Ross, 2009). The stochastic process that satisfies Brownian motion is denoted $\{Z\}$ and it goes to states $\{z\}$.

1.1.2.1 *Black-Scholes Option Pricing Model*

Black-Scholes option pricing model is the partial differential equation,

$$\frac{\partial f}{\partial t} + rs\frac{\partial f}{\partial s} + \frac{\sigma^2 s^2}{2}\frac{\partial^2 f}{\partial s^2} - rf = 0. \tag{1.1}$$

, where $s = S(t)$ is the price of the underlying asset at time t, $f = f(s,t) = C$ is the unknown price of an option of which the underlying price is s at time t, r is the risk-neutral continuously compounded nominal interest rate, and σ is the volatility (Black & Scholes, 1973).

1.1.2.2 *Binomial Model*

The binomial model for option pricing is based upon a special case in which the price of a stock over some period can either go up by u percent or down by d percent (Ross, 2009). If S is the current price then next period the price will be either $S_u = S(1+u)$ or $S_d = S(1+d)$. If a call option is held on the stock at an exercise price of E, then the payoff on the call is either $C_u = \max(Su - E, 0)$ or $C_d = \max(Sd - E, 0)$. Let the risk-free interest be r and assume $d < r < u$ (Hull, 2000). The equation

$C_0 = S_0 \sum_{i=a}^{N} \binom{N}{i} \overline{q}^{N-i} (1-\overline{q})^i - Ke^{-rT} \sum_{i=a}^{N} \binom{N}{i} q^{N-i} (1-q)^i = S_0 \mathcal{Q}_1 - Ke^{-rT}\mathcal{Q}_2$, $a = 1, 2, \cdots, N$, where $\overline{q} = uqe^{-r\Delta t}$, $q = \frac{e^{r\Delta t} - d}{u-d}$, $u = \frac{e^{r\Delta t} - d}{q} + d$ and T is maturity period, is a binomial model.

A binomial model can be used in a situation in which one wants to price an option for a motor vehicle fuel in whose value in each time period is its price in the increases by a rate say $u = 25\%$ and decreases by a rate say $d = 20\%$ in a situation in which the bank rate (risk-free rate) is say $r = 8\%$. American option can be more realistic in this case, explaining why a binomial model is needed (Bot et al., 2013).

1.1.2.3 Stochastic volatility model

The fact that options are used to hedge out risks shows that the risk-neutral assumption is unrealistic (Hull, 2000). Secondly, any Brownian motion occurs if the change in the underlying price has a Markov property (Ross, 2009). However, it has been shown by researchers that , in cases where risk can not be neglected, the volatility of underlying price tends to have long-term memory (So, 2002), that is, future volatility tends to depend on past ones. Thus, the second assumption is equally unrealistic.

This partially explains why the Black-Scholes model fails to accurately price options with an American style of exercising them. It does not consider the steps along the way where there could be possibility of early exercise of an American option (Kuo, 2002). To avoid this problem, one can use the binomial model that breaks down the time interval into potentially a large number of intervals or steps. Therefore it is more accurate than the Black-Scholes Model for an American Option.

However, owing to the Brownian motion and the risk-neutral assumptions, both of these models tend to under estimate option prices. In particular, the Binomial model has the disadvantage that it is very slow even with fastest of the modern day computers (Zariphopoulou, 2001). This has led many researchers to the inclusion of corrections to the option pricing model leading also to a class of models for optional pricing called stochastic volatility models that may be constructed through functions called utility functions (Grasselli & Hurd, 2007) or by simply including a risk aversion parameter (γ) in a model to quantify the degree of any attempt to reduce uncertainty taken by a holder of an option (Betteridge, 2005).

This is done through a process that is equivalent to Fourier series analysis of a partial differential equation. For example, the partial differential equation (1.1) can be transformed under Fourier series to include the stochastic volatility assumption to obtain the model (Fodya, 2007)

$$c_t = \frac{1}{2}\left(s^2\sigma^2 c_{ss} + 2b\rho s\sigma c_{sy} + b^2 c_{yy}\right) + \left(a - \frac{b\rho(\mu - r)}{\sigma}\right) c_y + \frac{\gamma}{2} b^2(1-\rho^2)(c_y)^2 - \frac{(\mu - r)^2}{2\gamma\sigma^2}, \tag{1.2}$$

where μ is the expected drift in price, ρ is the correlation between the volatility driving random variable y representing non-traded assets values and s. The variables a and b are constants, and their roles are explained in the methodology.

The main set back to using the stochastic volatility models is that of their being complicated or not leading to closed solutions under more realistic assumptions (Zariphopoulou, 2001). For example, the Black-Scholes model has closed solution under European option assumption (Ketcheson, 2011), but a stochastic volatility model that can be deduced from this model under the same assumption may not have a closed solution (Betteridge, 2005). Since accuracy matters, different numerical methods can be used to obtain solutions in such cases (Langtangen, 2012).

1.2 Efficiency and Operator Splitting Method

In numerical methods, efficiency means that a method is accurate, computationally realistic, cheap in computational resources (especially time), and strongly stable (Faou, 2011). In this study, efficiency means that a method gives numerically realistic solution while using cheap computational resources. The main computation resource being computer processing unit time. We adopt this definition because there is no closed solution to the partial differential equation (1.2) (Grasselli & Hurd, 2007), so we cannot expect an accurate solution but a computationally realistic one.

The second reason is that one can always use a stable operator splitting scheme (Estep & Ginting, 2008). For this to make sense, we explain the meaning of operator splitting technique. The operator splitting method is a divide-and-conquer strategy which involves decomposition of unwieldy systems of partial differential equations into simpler subproblems and treat them individually using specialized numerical algorithms (Chertock, 2010).

The decomposition can be done based on physical interpretations of the operators in a model, which is called differential splitting or based on the complexity of the algebraic operations which is referred to as algebraic splitting (Harwood, 2011). For example, to use operator splitting technique, one assumes that a time partial change in a quantity $\partial c/\partial t$ is a sum of operators say $\mathcal{L}_1$ and $\mathcal{L}_2$, which is the partial differential equation $\partial c/\partial t = \mathcal{L}_1 + \mathcal{L}_2$ (Huang et al, 2011). The operators, contribute differently to the partial change, and hence one can assign random or stochastic weights A and B to the operators to represent their contributions to the partial changes. This can allow for solving of the separate equations $A\partial c/\partial t = \mathcal{L}_1$, $B\partial c/\partial t = \mathcal{L}_2$, and $A + B = 1$, instead of the equation $\partial c/\partial t = \mathcal{L}_1 + \mathcal{L}_2$. It is always possible to chose a stable scheme for each sub-problem (Harwood, 2011).

1.3 Problem Statement

This research intends to improve the works of researchers Betteridge (2005) and Fodya (2007), who solved the stochastic volatility model for option pricing (1.2) using full model methods, by using operator splitting technique.

The two authors, as well as many others like So (2002), have solved the option pricing problem under the stochastic volatility assumption rather than the Black-Scholes model with less realistic assumptions nor the binomial model which is more realistic but has convergence problems. These researchers have solved this problem using partial differential equations via finite different schemes, Monte Carlo Methods of different kinds like the Bayesian modeling method.

The operator splitting method is a divide-and-conquer strategy which involves decomposition of unwieldy systems of partial differential equations into simpler subproblems and treat them individually using specialized numerical algorithms (Chertock, 2010). The decomposition can be done based on physical interpretations of the operators in a model, which is called differential splitting or based on the complexity of the algebraic operations which is referred to as algebraic splitting (Harwood, 2011). This clearly means that this method is dimension and assumption adaptive, because it is possible to obtain subproblems each having only one differential operator only (Langtangen, 2012).

Thus, one can use all the methods that have been adopted by some recent researchers, including those they have not been used in order to avoid violating the assumptions (Grasselli & Hurd, 2007). These can be applied in the subproblems rather than the whole problem, and then combine the solutions, a process called recoupling (Harwood, 2011). Recoupling requires high order skills so as to obtain solutions that can be compared with those from previous research works. The approach has been used mainly in natural sciences to solve Schrodinger equations and has proved to be fast and accurate (Langtangen, 2012).

1.4 Research Objectives

1.4.1 Main Objective

The main objective of the research is to explore the efficiency of operator splitting method for option pricing via a stochastic volatility model.

1.4.2 Specific Objectives

Specifically we would like to

1. solve the stochastic volatility model using the operator splitting technique.

2. develop a computer program to implement an algorithm for computer simulations of the solution of the option pricing model and

3. compare the results of the operator splitting technique with the results of the current methods in use.

1.5 Rationale

The operator splitting technique is a well-established branch of mathematics that also deserves more likelihood of it being applied in financial modeling. The method is easy if it is successfully applied to solve a partial differential equation modeling a phenomena (Ikonen & Toivanen, 2005). This is so because algebraic splitting is used to obtain simpler sub-problems each with a well known quadrature (Chertock, 2010). This is an advantage because choosing appropriate method ensures that convergence, consistency, and stability are checked within sub-problems. If appropriate recombination scheme is used, these qualities are amplified to the whole problem (Harwood, 2011). Therefore, operator splitting method is a computationally efficient method, which means realistic solutions are obtained with polynomial time complexity.

1.6 Scope of this Dissertation

The study's main contribution is the construction of an operator splitting method that is more efficiently applied to harder and more realistic stochastic volatility models. In Chapter 2 relevant literature is reviewed to clarify the problem being solved in the study. In Chapter 3, a theoretical overview explaining the feasibility of the method developed is explained in details. Chapter 4 presents the results of the numerical methods and relevant comparisons are given and explained in details. Finally, chapter 5 presents the conclusion and further work suggested by this work.

Chapter 2

LITERATURE REVIEW

2.1 Introduction

The option pricing models which arise from the stochastic volatility models are usually multidimensional partial differential equations possessing forms that cannot be expressed in terms of known elementary functions (Joshi , 2008). Therefore, researchers have usually used numerical methods to solve these models (Langtangen, 2012). These numerical methods have ranged from finite difference, Monte Carlo, finite element, midpoint, trapezoidal, to Euler-Lagrange methods of numerical integration (Ross, 2009), just to mention a few.

Some authors classify these methods into two groups: those that are dimension adaptive and those that are not. The Monte Carlo methods are dimension adaptive (Gomes & Michaelides, 2007). These methods have been used by researchers as well as traders of options to set prices of option in order to communicate to buyers at a right time (O' Sullivan , 2010). This is so because the closed form solutions of the stochastic volatility models are rare (Ikonen & Toivanen, 2004). However, the invention of computers and software has led to the faster availability of results (Kuo, 2002). Since the invention of the Black-Scholes option pricing model (Black & Scholes , 1973) represented by equation (1.1), the trade in options has increased. Hence, efficient (that is fast and computationally realistic) methods are still needed. A lot of research work has been published to take care of this need.

2.2 Classical Numerical Methods

The work of Fodya (2007), as noted earlier, uses the finite difference methods to solve differential equations emanating from the stochastic differential equations under stochastic volatility modeling up to two dimensions. It follows the works of other researchers like Betteridge (2004), and So (2002) which have adopted the use of different kinds of Monte Carlo simulations and the Bayesian modeling. The results of the finite difference are compared with the results of past works of Monte Carlo methods. It is noted that, in aspects of time complexity and accuracy, the method adopted is better than the Monte Carlo methods.

However, the work of Fodya (2007) does not extend to the third dimensions. This is so because of the fact the sparse grid methods require more subdivisions which increase exponentially with the increase in dimensions (Betteridge, 2005). As such, they break down by being slower in time and convergence complexities. This problem is called the curse of dimensionality (Fodya, 2007).

The Monte Carlo simulations, which are used in the research work of Betteridge (2005), preceding the work of Fodya (2007) to solve the problem, are free from the curse of dimensionality. They have been applied to solve the same problem in the third and higher dimensions, but not the lower dimensions. The results of the work are compared with those the midpoint and trapezoidal rules, which had been adopted by past researchers.

Monte Carlo methods are computationally realistic (Betteridge, 2005). At the time of research of Betteridge (2005), quasi-Monte Carlo methods which can be adapted the deterministic models had been invented.

The option pricing models can be considered to be both deterministic and stochastic (Kuo, 2002). This has been known to many researchers like So (2002). The research works before So (2002) work had brought the results that had shown that stochastic volatility models and empirical data demonstrate long term memory (Zariphopoulou, 2001). Hence, So (2002) work adopts Bayesian estimation via Monte Carlo Markov Chain sampling to take care of the long-term memory syndrome. Better results than those of the other Monte Carlo methods are reported in this work (So, 2002).

Despite the fact Monte Carlo methods are invariant to the curse of dimensionality, it has been known that Monte Carlo methods may not well work with empirical data that has small sample size and demonstrates long-term memory (Carassus & Rasonyi, 2002). They are also perform poorly in modeling the problem in lower dimensions (Fodya, 2007) or higher dimensions. In dimensions lower than 3 they tend to give computationally unrealistic solutions to partial differential equations for the option pricing model, which are prices of options. In dimensions higher than 3 they tend to be slower (Hozman & Tichy, 2016).

2.3 Hybrid Numerical Methods

This made researchers work hard towards improving the Monte Carlo methods. One attempt was made by Benth and Vos (2013). They accustomed the multivariate stochastic volatility models to capture energy market (fuel and gas market) features like price spikes, mean reversion, stochastic volatility, inverse leverage effect , and dependence of commodities through the use of Fourier Series analysis. This was done deliberately to ensure that the Monte Carlo simulations used led to efficiently computed results. The work had effectively incorporated ergodic assumption (Ross, 2009). This was so, because the second order structure, stationarity, and Gaussian drift assumptions were included simply to achieve convergence in the Monte Carlo methods (Carassus & Rasonyi, 2002). So the results were promising but not very different from those that could be computed from the usual Black-Scholes option pricing model (Ross, 2009).

This prompted Benth and Vos (2013) to consider eliminating the Gaussian drift assumption using Fourier Series analysis method to obtain a more realistic stochastic volatility model (Benth & Vos, 2013). Thus, a little retardation to convergence of Monte Carlo method had been introduced to their earlier work for accurate prices. More realistic solutions were obtained.

However, the solutions depicted positive forward humps in the option pricing (Benth & Vos, 2013), which made them not very consistent with standard volatility jumps in the energy market (Fodya, 2007). Thus, the results were good but more suited for forwards contracts pricing rather than option pricing (Hull, 2000). Anticipating this phenomenon, various researchers, like Takkabutr (2013), consider deriving risk aversion parameter from stochastic volatility with Hestons transformation under time series model regression, purely using least squares method while maintaining finite differences in the stochastic volatility model. The consequence of this is that computational and convergence time have increased with number of iterations in the finite differences , despite obtaining results that are computationally more realistic.

Working against this problem, many mathematical finance researchers like Du et al (2014) propose accelerating Monte Carlo methods for pricing multi-asset options under the stochastic volatility models developed by Hull and White (Hull, 2000). This improved the efficiency of variance reduction by obtaining more accurate prices of multi-asset options. However, it is noted in the same research work that the methods perform poorly when the option is out-of-the-money.

Lee (2014) developed a Monte Carlo engine for using a hybrid stochastic-local volatility model to price exotic options . This method performed poorly in Vanilla options, which are the most common options available on the options market. In addition, it is clearly noted in this paper that increasing strike price leads to increase in relative error. That is, the method is clearly not computationally efficient. In search for the reasons for failure of Monte Carlo methods, some researchers have always gone back to finite differences. For example, Hozman and Tichy (2016) investigate numerical damagingness of boundary conditions in particular the Dirichlet, Neumann and transparent boundary condition. They considered introducing to stochastic volatility model the finite difference method based discontinuous Galerkin criterion (Hozman & Tichy, 2016). This Improved speed of computation. However, it is also noted in the same research work that relative error increase with increase in strike price. The method can also break down at some strike values because of the discontinuous Galerkin criterion (Faou, 2011).

2.4 Need for Operator Splitting Technique

One way to explain the failures in the Monte Carlo and finite difference methods, is that most of the option pricing models contain at least two terms which are usually the diffusion (heat) term and the wave term. For example, in equation (1.1) the second term is the heat term and the third is the wave term, explaining why this equation would model cash flow (Langtangen, 2012).

If the partial differential equation contains the heat term and first term only, then forward Euler finite difference method will always be unstable for all Courant-Fredrick-Levy numbers that we can choose (Thalhammer, 2008). This means it would not converge (Langtangen, 2012). However, the leap-frog method with second order accuracy in space and time will be stable for all Courant-Fredrick-Levy numbers (Langtangen, 2012). Similarly, if the differential equation contained the wave term only, then the forward Euler implicit finite difference method would be stable while the leap-frog method will be unstable (Thalhammer, 2008). Since most option pricing models contain both the heat and wave term, then choosing one of these finite difference methods or one Monte Carlo method will not be justified dynamically (Harwood, 2011), that is in terms of the physical properties of the terms in the model.

This problem had been noted by researchers in other scientific fields like chemistry and engineering in which Schrodinger equations (Thalhammer, 2008), which is very similar to the Black-Scholes linear operator ,are used to solve physical problems like modeling of optimal lead acid battery compositions (Harwood, 2011). To solve these problems which usually contain the heat and wave terms (Lucas, 2008), a method of splitting the whole equation into separate and small equations each having one of the terms based on physical properties of the terms (dynamic splitting) or based on simplicity of the expressions (algebraic splitting) has been adopted.

2.5 Classical Operator Splitting Methods

These equations are solved separately using the most appropriate methods for each equation separately, and then the solutions are recombined to approximate the solution of the original partial differential equation (Huang et al, 2011). For example, in this problem one would split the differential equation and use implicit Euler equation and leap-frog method separately, and then combine the solutions to estimate the solution of the original problem. The most interesting work in areas such as this is that of Harwood (2011) in which the method is used in semi-linear parabolic equations modeling lead acid battery composition, which shows clearly that this method is as well dimension adaptive. It is noted in this paper that any form of operator splitting method outperforms a similar one applied in full method. That is, explicit Euler scheme are found to be accurate and faster in split method than in full method.

He managed to obtain a reliable optimal composition of lead acid battery using this method. These results were consistent with the results of a research work of Thalhammer (2008) three year earlier that deduced possibility of a high order error bound in operator splitting method for exponential splitting. Mechthild (2008) demonstrates his results for linear Schrodinger equations. He also numerically demonstrates that the splitting methods retain their convergence in the Schrodinger equation.

Working under similar motivation, Huang et al (2011) solve Maxwell's equation using optimized operator splitting in three dimensions for the first time. They use finite differences for each approximation in the split sub-problems. They are also able to increase efficiency through a high order approximation scheme. This makes them obtain numerical schemes with high order accuracy, and demanding low computational resources.

In the same year, other researchers , Jia et al (2011), developed a numerical scheme for solving the Boussinesq and the Navier- Stokes equations of inviscid flow based on operator splitting method. They used a combination of finite element method and Euler's theta method to solve the split sub-problems from the incompressible non-stationary thermal convection problems. The method was very efficient. In addition, a unique solution which was stable at different Reynold numbers was obtained.

Another independent researcher Faou (2011), in the same year, was able to derive hybrid numerical schemes based on Monte Carlo recombination of solutions of split sub-problems upon considering stochastic interpretation of split sub-problems of a reaction-diffusion problem. He obtained solutions and stability criteria that were very efficient but not available in research works before his work.

The main set back with operator splitting methods is that many researchers think that solutions tend to either under estimate or overestimate the solutions of the problem to be solved (Estep & Ginting, 2008). However, the works of many researchers like Chertock (2010) have shown that, by selecting a proper recombination scheme, it is possible to avoid this problem.

One attempt to solve indifference pricing models with stochastic volatility using the method is made by Ikonen and Toivanen (2004). The main focus of the work has been the one of solving the American Option pricing problems using simple models. One observation made in this research is that splitting does not increase the error of convergence, which was taking care of the problem of increase in relative error noted by recent researchers studied in this chapter. Ikonen and Toivanen (2005) compared their results with other full model methods and realized that the operator split methods were faster. Their results have not been different from the common results in other numerical methods which emerged after this research namely the works of Betteridge (2005) and Fodya (2007) obtained without a splitting method. The works of Ikonen and Toivanen (2005) do not consider efficient solution method for systems of linear and nonlinear parts equations arising from the option pricing models. However, the recent works of researchers like Estep and Ginting (2008) show that selection of efficient methods through systematic sampling of the split sub-problem, which is purely random and computer simulation based, leads to better results which are not similar to ordinary ones.

2.6 Improving Operator Splitting Schemes

Similarly, the authors Ikonen and Toivanen (2004) would have used the method of Li et al (2004) who are able to obtain a fast, robust, and accurate scheme of measuring crystal growth embedded in a phase-shift equation using operator splitting methods. Li et al (2004) split the governing phase-field equation into three parts. The first is calculated using explicit Euler method, the second by multigrid method, and the last part which is non-linear using a closed form solution rather than simply bull-dozing finite difference methods through out.

The main question probably posed by Ikonen and Toivanen (2004) that makes them fail to use proper method for the linear and non-linear part is that of what would happen if one used difference scheme which is iterative and then a closed form that is true on its own without iteration (Hozman & Tichy, 2016). Using iteration throughout would lead to creation of answers that are very far from the truth because the iteration in the closed form would lead to manipulation of true answers to something different (Langtangen, 2012). The answer to this question is partly given by the approach of Li et al (2011).

Another approach which the work in this thesis has adopted is that of maintaining iterative methods in the linear just like Li et al (2011), whilst using Runge-Kutta methods embedded in Strang's split-recombination method like Yazici (2010) who not only shown how to improve the efficiency of the classical operator splitting methods but also applied this idea to the KdV equation, and then applying the Charpit method suggested by Fodya (2007). The Runge-Kutta is maintained even in the Charpit method to allow for iterative method and maintain physical properties of the problem model just like the methods developed by Delgado (1997) and Ketcheson (2011).

2.7 Summary

In summary, different methods are devised to solve different problems. For our problem at hand, it has been shown that up to two dimensions, numerical partial differential equations are more efficient in time and error convergence than Monte Carlo methods. One can always use the Feynman-Kac formula to convert a stochastic differential equation into a partial differential equation (Fodya, 2007). The work of this research aims at using the operator splitting method to try to improve the already existing methods that use numerical partial differential equations.

Chapter 3

METHODOLOGY

3.1 Introduction

The methodology used in this study is desk review. Literature on option pricing, stochastic differential equations, partial differential equation , numerical methods, and operator splitting methods are analyzed. The operator splitting method was used to solve the partial differential equation from the stochastic volatility model for option pricing. Convergence criterion for each numerical algorithm assigned to a sub-problem was established. A proper re-coupling technique is selected by running permutations which maintain consistency of solutions. Matrix laboratory [1] programming language is used to achieve this.

Strang recombination scheme, under the fourth-order Runge-Kutta method was applied throughout. These were extended to the non-linear part of the stochastic volatility equation under Charpits method. The source of data was the financial data used in the previous research, Chicago Board of Option Exchange (CBOE) [2], and Yahoo Finance (YF) [3]. Focus was placed on jet fuel option prices data.

[1] www.mathsworks.org
[2] cboe.org
[3] www.yahoo.com/finance

The simulations were run together with those available in current research papers so that appropriate comparison of efficiency of the methods could be made in the light of empirical data. First, we would like to have a closer look at the common option pricing models introduced earlier. We focus at the call option.

3.1.1 Black-Scholes Model

The value of a call option on a nondividend paying stock can be dictated by a number of factors; the current price of the stock S, the exercise price X, the time until expiration t, the risk-free interest rate r, the volatility of the stock price $q = \sigma$, and the expected rate of return on the stock μ. Let C be the price of the call option. The functional dependence can then be expressed as: $C = C(S, X, t, r, q, \mu)$. The analysis will reveal that the last variable, μ, plays no role in determining option value for this case (Langtangen, 2012). The change in stock price dS is assumed to be given by: $dS = \mu S dt + qSdz$. Ito's Lemma (Hull, 2000) states that $dC = [(\partial C/\partial t) + (\partial C/\partial S)\mu S + (1/2)(\partial^2 C/\partial S^2)q^2 S^2]dt + (\partial C/\partial S)qSdz$. Now consider a portfolio containing one written call (whose value is $-C$) and h shares of the underlying stock (Ross, 2009). The value V of this portfolio is given as: $V = hS - C$. The change in value is then $\frac{dV}{dC} = h\frac{dS}{dC} - 1 \Rightarrow dV = hdS - dC$. If h is equal to $h = \dfrac{\partial C}{\partial S}$ then $\frac{dV}{dC} = \dfrac{\partial C}{\partial S}\frac{dS}{dC} - 1 \Rightarrow dV = \dfrac{\partial C}{\partial S}dS - dC$.

This means that the change in the value of the portfolio dV over the interval dt is: $dV = \dfrac{\partial C}{\partial S}(\mu S dt + qSdz) - [(\dfrac{\partial C}{S})\mu S + (\dfrac{\partial C}{\partial S}\dfrac{\partial C}{\partial t}) + (1/2)(\dfrac{\partial C}{\partial S}\dfrac{\partial^2 C}{\partial S^2})q^2 S^2]dt - (\dfrac{\partial C}{\partial S})qSdz$. When terms are combined we find that those involving dz cancel out. Also the terms involving μ cancel out leaving: $dV = [-(\dfrac{\partial C}{\partial t}) - (1/2)(\dfrac{\partial^2 C}{\partial S^2})q^2 S^2]dt$.

Thus V is independent of the random variable dz; that is a risk free portfolio (Betteridge, 2005). Also the value of dV is independent of the expected rate of return μ (which is also the expected rate of growth of stock price S). Since the value of the portfolio is independent of the random variable it should increase in value at the same rate as the risk free interest rate (Ross, 2009); in other words, $dV = rVdt = r[(\frac{\partial C}{\partial S})S - C]dt$. For this to hold for all dt requires that (Black & Scholes , 1973) $(\frac{\partial C}{\partial t}) + (1/2)(\frac{\partial^2 C}{\partial S^2})q^2S^2 = -r(\frac{\partial C}{\partial S})S + rC$, or $(\frac{\partial C}{\partial t}) + (\frac{\partial C}{\partial S})rS + (1/2)(\frac{\partial^2 C}{\partial S^2})q^2S^2 = rC$. This is the Black-Scholes model introduced earlier as equation (1.1). The left hand side is a linear operator.

3.1.2 Binomial model

The binomial model for option pricing is based upon a special case in which the price of a stock over some period can either go up by u percent or down by d percent (Ross, 2009). If S is the current price then next period the price will be either $S_u = S(1+u)$ or $S_d = S(1+d)$. If a call option is held on the stock at an exercise price of E then the payoff on the call is either $C_u = \max(Su-E, 0)$ or $C_d = \max(Sd-E, 0)$. Let the risk-free interest be r and assume $d < r < u$ (Hull, 2000).

Now consider a portfolio made up of one written call and h shares of the stock (Ross, 2009). That is to say, the owner of the portfolio owns h shares of the stock and then sells (writes) one call with an expiration date of one period. If the stock price goes up the portfolio has a value of $V_u = hS(1+u) - C_u$ and if it goes down $V_d = hS(1+d) - C_d$ (Fodya, 2007). Suppose h is chosen so that the portfolio has the same price whether the stock price goes up or goes down (Ross, 2009). The value of h that achieves this condition is given by $hS(1+u) - C_u = hS(1+d) - C_d$ or $h = \dfrac{C_u - C_d}{S_u - S_d} = \dfrac{\max(S_u - E, 0) - \max(S_d - E, 0)}{S_u - S_d}$.

Thus, the ratio h can be determined. In particular, it does not depend upon the probability of a rise or fall (Hull, 2000). The value of h that make the value of the portfolio independent of the stock price is called the hedge ratio. A portfolio that is perfectly hedged is a risk-free portfolio so its value should grow at the risk-free rate r.

The current value of the hedged portfolio is the value of the stocks less the liability involved with having written the call (Ross, 2009). If C represents the value of owning the call then the liability involve with having written the call is $-C$. Therefore the value of the portfolio is $(hS - C)$. After one period of growing at the risk-free rate its value will be $(1+r)(hS - C)$, which is the same as $(hS(1+u) - C_u) = (hS(1+d) - C_d)$. Solving for C gives $C = hS - (hS(1+u) - C_u)/(1+r) = hS - hS(1+u)/(1+r) + C_u/(1+r) = hS[1 - (1+u)/(1+r)] + C_u/(1+r) = (hS(r-u) + C_u)/(1+r) = [-hS(u-r) + C_u]/(1+r)$ If $(r-d)/(u-d)$ is denoted as p and h is eliminated, then $C = [pC_u + (1-p)C_d]/(1+r)$ Using the same recursive arguments for any $n \in N$ time steps, the following general valuation binomial option pricing formula can be obtained (Ross, 2009).

$C_0 = e^{-rT} \sum_{i=0}^{N} (S_0\, u^{N-i}\, d^i - K)^+ \binom{N}{i} q^{N-i} (1-q)^i$

$= e^{-rN\Delta t} \sum_{i=a}^{N} (S_0\, u^{N-i}\, d^i - K) \binom{N}{i} q^{N-i} (1-q)^i$

$= S_0 \sum_{i=a}^{N} \binom{N}{i} (u\, q\, e^{-r\Delta t})^{N-i}\, (d\, e^{-r\Delta t}\, (1-q))^i - Ke^{-rT} \sum_{i=a}^{N} \binom{N}{i} q^{N-i} (1-q)^i$

$= S_0 \sum_{i=a}^{N} \binom{N}{i} \overline{q}^{N-i}\, (1-\overline{q})^i - Ke^{-rT} \sum_{i=a}^{N} \binom{N}{i} q^{N-i} (1-q)^i$

$= S_0\mathcal{Q}_1 - Ke^{-rT}\mathcal{Q}_2$, where $\overline{q} = uqe^{-r\Delta t}$, $q = \frac{e^{r\Delta t}-d}{u-d}$. Then, $u = \frac{e^{r\Delta t}-d}{q} + d$. Therefore,

$1 - \bar{q} = 1 - uqe^{-r\Delta t} = 1 - \big(e^{r\Delta t} - d\big)e^{-r\Delta t} - dqe^{-r\Delta t} = de^{-r\Delta t} - dqe^{-r\Delta t} = de^{-r\Delta t}(1-q)$ (Ross, 2009).

3.2 Operator Splitting Method

The change in the price of non-risky commodity is directly dependent on the risk-free interest rate (r_t) and the price (C_t) itself. Mathematically, we write $dC_t = r_tC_tdt$. An investor is to pay out for a contract at time T and amount $P = P(S_t, Y_t)$, which depends on stock price S_t and the price of non-traded assets Y_t which yield volatility σ. This study wishes to solve

$$c_t + \frac{1}{2}\left(s^2\sigma^2 c_{ss} + 2b\rho s\sigma c_{sy} + b^2 c_{yy}\right) + \left(a - \frac{b\rho(\mu - r)}{\sigma}\right)c_y + \frac{\gamma}{2}b^2(1-\rho^2)(c_y)^2 - \frac{(\mu-r)^2}{2\gamma\sigma^2} = 0 \tag{3.1}$$

subject to $c(T, s, y) = P(s, y)$, $\forall\ s \in [0, S]$ and $\forall y \in [0, Y]$. Details of this formulation are found in (Grasselli & Hurd, 2007). Most of the letters in the Heston's model (3.1) are defined in the earlier sections of this chapter under the discussions of Binomial and Black-Scholes model. The letters a and b are scale variables (Fodya, 2007), ρ is the correlation between s and y. The symbol γ is the risk averse parameter of the exponential utility function of the form $-e^{-\gamma t}$. Consider the optional pricing model under stochastic volatility of the form: (3.1). We split this into 5 split sub-problems and apply the Strang algorithm as follows

$Ac_t = \frac{1}{2}s^2\sigma^2 c_{ss},\ t \in\ [t_0, t_1],\ c(t_0) = c_0$

$Bc_t = \frac{1}{2}b^2\sigma^2 c_{yy},\ t \in\ [t_1, t_2],\ c(t_1) = c(t_{\frac{1}{2}})$

$Cc_t = b\rho s\sigma\sigma^2 c_{sy},\ t \in\ [t_2, t_3],\ c(t_2) = c(t_{\frac{3}{2}})$

$Dc_t = \left(a - \frac{b\rho(\mu-r)}{\sigma}\right) c_y,\ t \in\ [t_3, t_4],\ c(t_3) = c(t_{\frac{5}{2}})$

$Ec_t = \frac{\gamma}{2} b^2(1-\rho^2)(c_y)^2 - \frac{(\mu-r)^2}{2\gamma\sigma^2},\ t \in\ [t_4, t_5],\ c(t_4) = c(t_{\frac{7}{2}})$

$Aq_t = \frac{1}{2} s^2\sigma^2 q_{ss},\ t \in\ [t_{\frac{7}{2}}, t_5],\ q(t_{\frac{7}{2}}) = q(t_5)$

$A + B + C + D + E = 1$

where q is some weighting function corresponding to the option price c, and letters $\{A,\ B,\ C,\ D, E\}$ are random weights representing the contributions of the operators to the time partial change in option price.

It can be noted that under central differences approximation to partial derivatives the sub-problems can be expresses as $A\frac{1}{2\Delta t}\langle 1,\ -1\rangle = \frac{1}{2}s^2\sigma^2\frac{1}{4\Delta s^2}\langle 1,\ -2,\ 1\rangle$, $B\frac{1}{2\Delta t}\langle 1,\ -1\rangle = \frac{1}{2}b^2\sigma^2\frac{1}{4\Delta y^2}\langle 1,\ -2,\ 1\rangle$, and $C\frac{1}{2\Delta t}\langle 1,\ -1\rangle = b\rho s\sigma\sigma^2\frac{1}{2\Delta s\Delta y}\langle 1,\ -1,\ -1,\ 1\rangle$. The finite differences through Runge-Kutta method for the first four sub-problems and the Charpit's method in the last sub-problem were applied. This chapter presents theoretical feasibility of the method which also includes consistency, stability and error analysis.

3.3 Consistency Analysis

Definition 3.3.1. *An operator splitting method is* ***consistent*** *if it satisfies the following conditions:* $\lim_{h\to 0}\sup_{[0,T-h]} \dfrac{\|(V(h)-\Phi(h))(c)\|}{h} = 0$ *and* $\dfrac{\|(V(h)-\Phi(h))(c)\|}{h} = \mathcal{O}(h^p)$*, where* $\mathcal{O}(h)$ *is the truncation error under Taylor series approximation operated by* h*,* p *is a real number,* V *is the* ***exact solution operator*** *and* Φ *is the* ***approximate solution operator*** *(Carassus & Rasonyi, 2002).*

Definition 3.3.2. ***Definition of Fourier transform*** *A Fourier transform of* *a function f is defined as* $\hat{f}(\lambda) = \int_{\infty}^{\infty} f(t)\exp(-i\lambda t)dt$, *where* $i = \sqrt{-1},\ \lambda \in \mathbb{R}|\{0\}$ *(Langtangen, 2012).*

The operators $A\frac{1}{2\Delta t}\langle 1,\ -1\rangle = \frac{1}{2}s^2\sigma^2\frac{1}{4\Delta s^2}\langle 1,\ -2,\ 1\rangle$, $B\frac{1}{2\Delta t}\langle 1,\ -1\rangle = \frac{1}{2}b^2\sigma^2\frac{1}{4\Delta y^2}\langle 1,\ -2,\ 1\rangle = \left(\frac{\frac{1}{2}b^2\sigma^2}{4\Delta y^2}\right)(c_{i+1}^n - 2c_i^n + c_{i-1}^n)$, and $C\frac{1}{2\Delta t}\langle 1,\ -1\rangle = b\rho s\sigma\sigma^2\frac{1}{2\Delta s\Delta y}\langle 1,\ -1,\ -1,\ 1\rangle$ are bounded. We verify the boundedness of the operators. Proofs of most of the propositions, and lemmas so quoted and applied are not replicated in this section. If we let $\dfrac{1}{4\Delta y^2}\langle 1, -2, 1\rangle = \partial_y^+\partial_y^- c_i^j$, where $\partial_y^- c_i^j$ backward difference approximation to partial derivative and $\partial_y^+ c_i^j$ is foward difference approximation to partial derivative, then the explicit Euler scheme applied to the first, the second, and the sixth split sub-problems can be defined as $A_w\partial_t^+ c_i^j = C_w\partial_y^+\partial_y^- c_i^j$ together with the boundary in (3.1) conditions and $c_S^T = c_Y^T = c_0^0$ (Fodya, 2007) while the implicit euler scheme is defined as $A_w\partial_t^- c_i^{j+1} = C_w\partial_y^+\partial_y^- c_i^{j+1}$ under the same boundary conditions (Carassus & Rasonyi, 2002). In this case $A_w \in \{A, B\} \subset \mathbb{R}$ and $C_w \in \left\{\frac{1}{2}s^2\sigma, \frac{1}{2}b^2\sigma^2\right\} \subset \mathbb{R}$. We need A_w to be a state of a stochastic process, and hence, it is determined randomly (Ross, 2009).

Suppose c_i^j is a solution of the first, the second, and sixth split sub-problems, then using the argument considered by Dimarco and Pareschi (2011), and maintaining the boundary conditions considered in problem (3.1) the explicit Euler method will satisfy $\sup\limits_{\forall i}\|c_i^T\| \leq \sup\limits_{\forall i}\|c_i^0\|$ provided the Courant-Fredrick-Levy number $\left(\frac{C_w\Delta t}{4A_w\Delta y^2}\right)$ is within the interval $[0, 1]$. For the implicit Euler scheme, the same condition is satisfied but for all Courant-Fredrick-Levy numbers (Carassus & Rasonyi, 2002). In this paper as noted in the next section the Courant-Fredrick-Levy number cannot be zero, otherwise the physical meaning in equation (3.1) is altered

(Chockalingam & Muthuraman , 2011). Thus the operators in the first and the second split sub-problems are bounded (Yazici, 2010).

Similarly, the difference equation endowed in the third split sub-problem can be expressed as $A_w \partial_t^- c_i^{j+1} = C_w \partial_y^+ \partial_s^- c_i^{j+1}$, where $A_w = C \in \mathbb{R}$ and $C_w = b\rho s\sigma^2$. If c_i^j satisfies this sub-problem, we have $\sup_{\forall i} \|c_i^T\| \leq \sup_{\forall i} \|c_i^0\|$ for all non zero Courant-Fredrick-Levy numbers $\left(\dfrac{C_w \Delta t}{4A_w \Delta s \Delta y}\right)$ with respect to analysis in (Faou, 2011). Thus the operator in the third split sub-problem is bounded.

The fourth split sub-problem clearly satisfies the same conditions because $|A_w \partial_y^+ c_j - C_w c'(y_j)| \leq 0.5 \Delta y^2 |(A_w - C_w)c_j|_{C^3([0,Y]\times[0,T])}$. This analysis is done by using Taylor series and is similar to the one applied by (Fowler & Winstanley, 2012).

If we allow $c = c(t,s,y) = a_1 t + a_2 y + a_3 s + a_4 = a_1 t + a_2 y + a(s)$, $\{a_i : i = 1,2,3,4\} \subset \mathbb{R}$ to be the solution of the non-linear sub-problem, then the solution has the form $c = \left(D_w a_2^2 + \frac{(\mu-r)^2}{2\gamma\sigma^2}/E\right) t + a_2 y + a(s)$, where $D_w = \dfrac{\frac{\gamma}{2} b^2(1-\rho^2)}{E}$, $c_t = a_1$, $c_y = a_2$. This is an exact solution under Charpit method (Delgado, 1997). To approximate this solution we adopt the following iteration: $k_2 = \dfrac{(u-r)^2}{2\sigma^2(y_j)}$, $k_1 = \frac{\Delta t}{E \Delta y}\left(c_j^{n+1} - \dfrac{(u-r)^2}{2\sigma^2(y_j)} s_j\right)$, $k_3 = c_{j-1}^{n+1} - \left(y_j + s_j \sqrt{\frac{s_j}{k_2}}\right) + k_1$, $c_{j+1}^{n+1} = y_j + s_j . \sqrt{\frac{s_j}{k_2}} + k_3$, which is similar to the method adopted by Geng (1993) and Takkabutr (2013). From the analysis in the previous paragraph, the coefficients (a_1 and a_2) are bounded.

It follows from this analysis that the split sub-problems used in this study agree up to $\mathcal{O}(\Delta t^3)+\mathcal{O}(\Delta y^3)$. In other words, if we let V_{123456} and Φ_{123456} be exact and approximate solutions of the method used in this study for the six sub-problems, respectively, then from Taylor series and combination of results similar to the one adopted by Lucas (2008) and Harwood (2011), we have $\|V_{123456}-\Phi_{123456}\| = \mathcal{O}(\Delta t^3; \Delta y^3)$.

3.4 Error Analysis and Accuracy

This subsection mainly explains how we arrive at this result in the last subsection. We consider the time part only.

To investigate the splitting techniques accuracy, the solution is solved over many small steps sizes allowing for $t_5 = \Delta t$. The accuracy of the algorithm used in splitting outlined above is is determined by the order under Taylor expansion (Rosencrans, 1972). One way determination is considered here because it has been shown that permutation of the same recombination system does not affect order of truncation error (Harwood, 2011).

Since we adopt the Strang operator splitting technique, the first and the second split sub-problem can be expressed as $(A+B)c_t = (\beta_1 L_1 + \beta_2 L_2)c$, where L_1 and L_2 are independent of s and x. The exact solution after time $\delta t = t_5$ to this is $c = e^{(L_1+L_2)\delta t}u_0 \approx e^{\frac{1}{2}L_1\delta t}e^{L_2\delta t}e^{\frac{1}{2}L_1\delta t}u_0$ (Lucas, 2008). Since L_1 operates the partial derivative in the direction of x, the operator $e^{L_1\delta t}$ just multiplies by $e^{L_2(x)\delta t}$. Since $L_1 = F\hat{L}_1F^{-1}$ is orthogonally diagonalizable by a Fourier transform F (Jia et al., 2011), assuming the boundary conditions in the problem (3.1) , $e^{L_1\delta t} = Fe^{\hat{L}\delta t}F^{-1}$, where $e^{\hat{L}\delta t}$ is the operator that multiplies each Fourier mode by $e^{-k^2\delta t}$.

Using similar definitions of exact solution under Strang splitting and the approaches followed by Harwood (2011), Jia et al (2011) and Lucas (2008), we consider the whole problem (3.1) as follows: $|c_{exact}(\Delta t) - \Phi(\Delta t)| = |L_1 \sin(\pi\Delta t)e^{L_1\pi^2\Delta t} - e^{\Delta t L_1}c^n + L_3\sin(\pi\Delta t)e^{L_3\pi^2\Delta t} - e^{\Delta t L_3}c^{n+3/2} + L_6\sin(\pi\Delta t)e^{L_6\pi^2\Delta t} - e^{\Delta t L_6}c^{n+1} + e^{L_4\Delta t}c^{n+5/2} - re^{\frac{\Delta t}{2}L_4}c^{n+5/2} + L_2\sin(2\pi\Delta t)e^{\Delta t} - re^{L_2\Delta t}c^{n+3/2} + rL_5^{-1}\Delta t - e^{rL_5^{-1}\Delta t}| = |(I + \Delta t L_1 c^n + \Delta t L_2 c^n + \cdots)(\pi\Delta t - \pi^2\Delta t^2/2 + \cdots) - \cdots| \propto |\frac{\Delta t^2}{2}\left(\sum \frac{\partial^i c^n}{\partial t^i}L_i c^n - \sum L_i c^n + \cdots\right) + \mathcal{O}(\Delta t^3)| = \mathcal{O}(\Delta t^2)$

It follows that the algorithm has truncation error of order $\mathcal{O}(\Delta t^2)$ based on agreement of order of agreement of the sub-problems (Carassus & Rasonyi, 2002).

3.5 Stability Analysis

It is imperative to determine the Courant-Friedrichs-Levy condition for the stability of the explicit solution of the PDE using the Von Neumann stability analysis . A very versatile tool for analysing stability is the Fourier method developed by Von Neumann. Now the Strang algorithm can be represented as $\frac{\partial}{\partial t}C* = L * v*$, $C* = [(C*)_{ij}]_{1\times 6}$, $L* = [(L*)_{ij}]_{6\times 6}$, $v* = [(v*)_{ij}]_{1\times 6}$, where $(C*)_{11} = Ac$, $(C*)_{12} = Bc$, $(C*)_{13} = Cc$, $(C*)_{14} = Dc$, $(C*)_{15} = Ec$, $(C*)_{16} = Aq$, $(L*)_{ij,\ i\neq j} = 0$, $(L*)_{11} = \frac{1}{2}s^2\sigma^2 L_1$, $(L*)_{22} = \frac{1}{2}b^2\sigma^2 L_2$, $(L*)_{33} = b\rho s\sigma\sigma^2 L_3$, $(L*)_{44} = \left(a - \frac{b\rho(\mu-r)}{\sigma}\right)L_4$, $(L*)_{55} = L_5$, and $(L*)_{66} = \frac{1}{2}s^2\sigma^2 L_6$. The fifth split sub-problem has non-linear operator $v_5 = (c_y)^2$ and hence L_5 is the algebraic operator on on v_5, $L_1 = \frac{\partial^2}{\partial s^2} = L_6$, $L_2 = \frac{\partial^2}{\partial s\partial y}$, $L_3 = \frac{\partial^2}{\partial y^2}$, and $L_4 = \frac{\partial}{\partial s}$.

3.5.1 Fifth Sub-problem

The stability of the fifth problem can be approximated as by letting $c(t, s, y) = \Delta y c(0, 0, y)y + \Delta s c(0, s, 0)s$, because Charpit's method is used under Runge-Kutta method in the Strang's splitting method adopted. Hence, applying the Fourier transformation gives $\dfrac{\hat{c}^{n+1}e^{i\lambda+\lambda} - \hat{c}^{n+1}e^{i\lambda-\lambda}}{\Delta t} = r\dfrac{\hat{c}^{n}e^{2i\lambda}}{2} - r_2 e^{i\lambda}$, from which $\rho(\lambda) = \dfrac{1}{1 + irr_2(e^{-3\lambda} - e^{\lambda})^2 \sin(\lambda)}$, which also implies that the system is stable when $r \neq 0$. In this case $r = \frac{\Delta t}{E\Delta y}$ and $r_2 = \frac{\Delta t}{E\Delta y}\dfrac{(u-r)^2}{2\sigma^2(y_j)}s_j$. The modulus of the product of rho's is clearly within unity, which implies that the whole recombination scheme is stable (Harwood, 2011) under the condition that the CFL numbers for each split sub-problem is not zero, which also implies that the parameters should not be zero.

3.5.2 Fourth sub-problem

Clearly, this is heat equation or one way wave equation. We can express this as $c_t = kc_x$, where $k = \dfrac{\sigma a - b\rho(\mu - r)}{D\sigma}$. Clearly, $e^{ix} = \cos(x) - i\sin(x)$ and $e^{-ix} = \cos(x) + i\sin(x)$, so that $2i\sin(x) = e^{-ix} - e^{ix}$, $i = \sqrt{(-1)}$. We are going to use this idea to establish the stability of the methods so used for each sub-problem.

Consider the scheme (Langtangen, 2012) $c_i^{j+1} = \begin{cases} c_i^j - k\frac{\Delta t}{\Delta x}\left(c_i^j - c_{i-1}^j\right), & k > 0 \\ c_i^j - k\frac{\Delta t}{\Delta x}\left(c_{i+1}^j - c_i^j\right), & k < 0 \end{cases}$
Since $k \neq 0$, then we don't want the parameters to be zero for the sake of fair comparison with classical methods (Grasselli & Hurd, 2007). In order to apply the Von Neumann stability analysis, we consider $\rho(\lambda) = \frac{\hat{c}^{n+1}(\lambda)}{\hat{c}^{n}(\lambda)}$, where $\hat{c}_i^n(\lambda) = e^{i\lambda x_i}$,

and obtain the following results, upon substituting in the scheme

$$\hat{c}^{n+1}(\lambda) = \begin{cases} \left(1 - k\dfrac{\Delta t}{\Delta x}\left(1 - e^{-i\lambda\Delta x}\right)\right)\hat{c}^n \ , & k > 0, \ j = n \\ \left(1 - k\dfrac{\Delta t}{\Delta x}\left(-1 + e^{i\lambda\Delta x}\right)\right)\hat{c}^n \ , & k < 0, \ j = n \end{cases}$$

From these results, we realise that, for $k > 0$, $\rho(\lambda) = \frac{\hat{c}^{n+1}(\lambda)}{\hat{c}^n(\lambda)} = 1 - k\dfrac{\Delta t}{\Delta x}\left(1 - e^{-i\lambda\Delta x}\right)$. It is clear, that the system is stable given that $0 < k \leq \dfrac{\Delta x}{\Delta t}$ (Yazici, 2010). We demonstrate this, by finding the condition that can be satisfied for $|\rho(\lambda)| \leq 1$, which is the condition for stability (Ketcheson, 2011). Using the identities $e^{ix} = \cos(x) - i\sin(x)$ and $e^{-ix} = \cos(x) + i\sin(x)$, so that $2i\sin(x) = e^{-ix} - e^{ix}$, $i = \sqrt(-1)$, we can rewrite this equation as $\rho(\lambda) = \frac{\hat{c}^{n+1}(\lambda)}{\hat{c}^n(\lambda)} = (1 - \alpha + \alpha\cos(\beta)) - i\alpha\sin(\beta)$, $\alpha = k\dfrac{\Delta t}{\Delta x}$, $\beta = -\lambda\Delta x$. From this, we realise that $|\rho(\lambda)|^2 = (1-\alpha)^2 + 2(1-\alpha)\alpha\cos(\beta) + \alpha^2$. Suppose $0 \leq \alpha \leq 1$, then $|\rho(\lambda)|^2 = (1-\alpha)^2 + 2(1-\alpha)\alpha\cos(\beta) + \alpha^2 \leq (1-\alpha)^2 + 2(1-\alpha)\alpha + \alpha^2 = 1$. However, $0 \leq \alpha \leq 1$ means $0 < k \leq \dfrac{\Delta x}{\Delta t}$. So in programming, we made sure that whenever this sub-problem was being solved under this scheme, this condition was satisfied. When $k < 0$, we realise that $\rho(\lambda) = (1+\alpha)^2 - 2\alpha(1+\alpha)\cos(\beta) + \alpha^2$. Suppose $-1 \leq \alpha \leq 0$, then $\rho(\lambda) = (1+\alpha)^2 - 2\alpha(1+\alpha)\cos(\beta) + \alpha^2 \leq (1+\alpha)^2 - 2\alpha(1+\alpha) + \alpha^2 = 1$. Hence, we choose $-\dfrac{\Delta x}{\Delta t} \leq k < 0$ in the programming to avoid instability problems. That is D was estimated through $\dfrac{\sigma a - b\rho(\mu - r)}{\text{rand}\left(\left(0, \dfrac{\Delta x}{\Delta t}\right] \cup \left[-\dfrac{\Delta x}{\Delta t}, 0\right)\right)\sigma}$, where the coefficient of volatility in the denominator means random number in either of the sets in the operation. The choice depends on the orientation of k. The details are given in the program in the Appendix B.

3.5.3 First and Second Sub-problems

We consider the scheme below that represent both sub-problems

$$A_w \frac{c_j^{n+1} - c_j^n}{\Delta t} = C_w \frac{c_{j+1}^n - 2c_j^n + c_{j-1}^n}{(\Delta x)^2} \tag{3.2}$$

because the first, the sixth, and the second split sub-problems are similar. In this case $A_w \in \{A, B\} \subset \mathbb{R}$ and $C_w \in \left\{\frac{1}{2}s^2\sigma, \frac{1}{2}b^2\sigma^2\right\} \subset \mathbb{R}$. If we apply similar analysis applied in the Subsection 3.5.2, we obtain the Courant-Fredrick-Levy condition $0 < \Delta t \leq \frac{(\Delta x)^2 A_w}{2C_w}$ (Langtangen, 2012). This limits our choice of the random weights A_w for splitting method. We adopt the following scheme to increase the range of choice of the random weights (Ketcheson, 2011) and to improve the scheme represented by Equation (3.2)

$$c_j^{n+1} - c_j^n = \frac{\Delta t C_w}{A_w(\Delta x)^2}\left(\frac{c_{j+1}^{n+1} + c_{j+1}^n}{2} - 2\frac{c_j^{n+1} + c_j^n}{2} + \frac{c_{j-1}^{n+1} + c_{j-1}^n}{2}\right).$$

Thus, we move the partial derivatives to time $n + \frac{1}{2}$ (Ketcheson, 2011) using averages. This matches with the splitting methods suggested above (Yazici, 2010).

In order to apply the Von Neumann analysis in the rewritten form of the scheme, we let $\hat{c}_j^n = \xi^n e^{i\lambda j \Delta x}$, where ξ represents the time dependence of the solution, $i = \sqrt{(-1)}$ and the exponential represents the spatial dependence of the solution. In the exponential $j\Delta x$ represents the position along the grid, and λ represents wave number. This is so because the split sub-problems are simply Diffusion Equations. Substituting this form, $\hat{c}_j^n = \xi^n e^{i\lambda j \Delta x}$, into the improved scheme we obtain $\xi - 1 = \frac{\Delta t C_w}{A_w(\Delta x)^2}\left(\xi e^{i\lambda\Delta x} + e^{i\lambda\Delta x} - 2\xi - 2 + \xi e^{-i\lambda\Delta x} + e^{-i\lambda\Delta x}\right)$.

Applying the identities $e^{ix} = \cos(x) - i\sin(x)$ and $e^{-ix} = \cos(x) + i\sin(x)$, we obtain $\xi = 1 - \frac{\Delta t C_w}{A_w(\Delta x)^2}(\xi + 1)[1 - \cos(\lambda\Delta x)]$. Expressing ξ explicitly, we obtain $\xi = \frac{1-\frac{\Delta t C_w}{A_w(\Delta x)^2}[1-\cos(\lambda\Delta x)]}{1+\frac{\Delta t C_w}{A_w(\Delta x)^2}[1-\cos(\lambda\Delta x)]}$. Clearly, the numerator is smaller than the denominator. Therefore, $|\xi| < 1$, which satisfies the stability criterion when $C_w \neq 0$. In programming $A_w = \frac{2\Delta t C_w}{(\Delta x)^2}$ was sufficient with respect to the above analysis. Hence, the methods in the first, second and the sixth split sub-problem are conditionally stable.

3.5.4 Third Sub-problem

We adopt the following numerical scheme for the third sub-problem (Fodya, 2007) $C\frac{c_{i,j}^{n+1}-c_{i,j}^{n}}{\Delta t} = 2b\rho s\sigma\frac{c_{i+1,j+1}^{n+1}-c_{i+1,j-1}^{n+1}-c_{i-1,j+1}^{n+1}+c_{i-1,j-1}^{n+1}}{4\Delta x\Delta t}$. Letting $k = \frac{2b\rho s\sigma\Delta t}{4C\Delta x\Delta t}$ and substituting, $\hat{c}_j^n = \xi^n e^{i\lambda j\Delta x}$, into the scheme we obtain $\xi - 1 = 2k\xi\left(e^{i\alpha} - e^{-i\alpha}\right)$, where $\alpha = \lambda\Delta x\Delta t$. Applying the identities $e^{ix} = \cos(x) - i\sin(x)$ and $e^{-ix} = \cos(x) + i\sin(x)$, we obtain $\xi - 1 = -4ik\xi\sin(\alpha)$. Making ξ subject of the formula we obtain $\xi = \frac{1}{1+4ik\sin(\alpha)} = \frac{1-4ik\sin(\alpha)}{1+16k^2\sin^2(\alpha)}$. From this we have $|\xi|^2 = \dfrac{1}{1+16k^2\sin^2(\alpha)} \leq 1$ Thus, the scheme is unconditionally stable.

3.6 Runge-Kutta (Numerical Method)

This section discusses Runge-Kutta method used in the operator splitting technique in this study. The fourth order Runge-Kutta method so adopted follows the following algorithm: $c_0 = \alpha$, $k_1 = hf(y_i, c_i)$, $k_2 = hf(y_i + \frac{h}{2}, c_i + \frac{1}{2}k_1)$, $k_3 = hf(y_i + \frac{h}{2}, c_i + \dfrac{1}{2}k_2)$, $k_4 = hf(y_{i+1}, c_i + \frac{1}{2}k_3)$, where $h = \dfrac{Y}{N} = \dfrac{S}{N} = \dfrac{T}{N}$ is the step size, and $f(y,c) = \partial_{x_i}^{orientation} c(T_j, y, s)$. Some researchers like Geng (1993), Yazic (2010), and Fowler & Winstanley (2012) have used this technique to solve the Schrodinger or Boltzmann equation in fluid mechanics.

It has been shown by a good number of researchers (Fowler & Winstanley, 2012) that this method has a truncation error of order $\mathcal{O}(h^4)$. As seen here, the definition of $f(y, c)$ is a partial derivative in difference formula form. This is done because we are approximating from a partial derivative not an ordinary one, and because we want to maintain the physical meaning of the differential equation, in order to obtain more realistic option prices. This is similar to the algorithm used by Yazic (2010) to solve the KDv (Korteweg–de Vries) equation in his paper.

3.7 Matrix Laboratory Programming

The Matrix Laboratory commands used in this programming made minimum use sparse matrices and hard coded solvers. It used simple commands to construct finite difference representation and the Runge-Kutta Method. Direction of evaluation within a matrix are changed withing the matrix to represent change in the direction of partial differentiation (Yazici, 2010). The Runge-Kutta scheme samples from the values of the partial derivatives. Special Matrix Laboratory technique is used to sample values along a line with particular value(s) of σ and y in order to compare the results with empirical data from the sources outlined in the first paragraph of this chapter. Details of the programming are given in the appendix B.

Chapter 4

RESULTS AND DISCUSSIONS

The numerical analysis is done on the scheme in the last chapter in the interval $[0, 1]$ for both stock price price and volatility random variable states. For comparison purposes the values in (Fodya, 2007) and (Betteridge, 2005) which are a $= 0.5$; b $= 0.3$; $\rho = 0.5$; $\gamma = 1$; $\mu = 0.04$; $r = 0.02$; $Y = 1$; $K = 0.8$; $\sigma(y) = 0.25 + \dfrac{1}{2\pi}\tan^{-1}(\lambda(y - 0.5))$ are maintained. This is done to maintain consistency, stability, and accuracy verified in chapter 3.

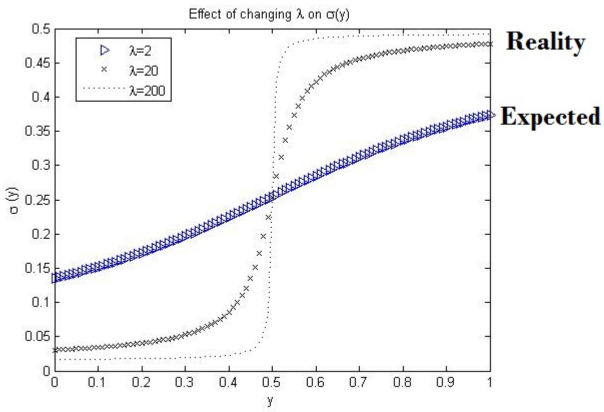

Figure 4.1: Standard Volatility Function

The empirical results on the volatility $\sigma(y) = 0.25 + \dfrac{1}{2\pi}\tan^{-1}(\lambda(y - 0.5))$ due to the changing jump parameter λ when the payoff is $(s - K)^+$ are similar to those evaluated by other researchers (Fodya, 2007; Grasseli & Hurd, 2007). The values of lambda are $\lambda = 2,\ 20,$ or 200. The results in figure 4.1 show that if the jump increases, the volatility jumps under the same payoff, and the curve steepens in the middle. Similarly, the curve flatten in the middle as the jump parameter lowers and the volatility values lower. This is one way to check the behavior of the option prices given by solution of stochastic volatility in terms of computational efficiency of method of approximating the solution.

4.1 Exploratory Numerical for Consistency

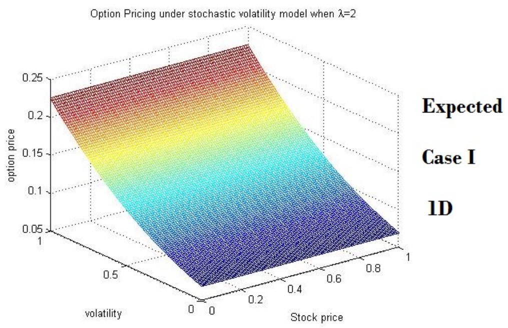

Figure 4.2: Option Prices in one Dimension Through OSM

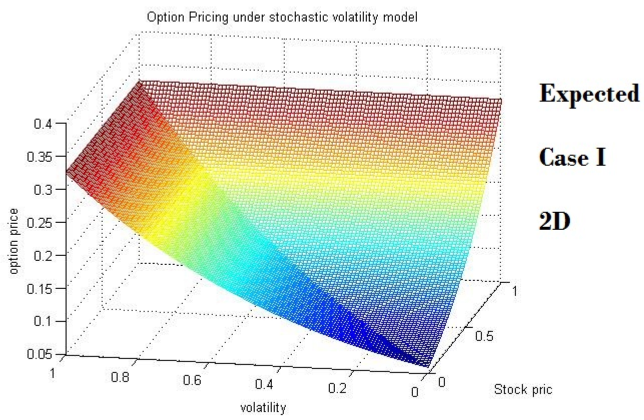

Figure 4.3: Option Prices in Two Dimensions Through OSM when $\lambda = 2$

We first run the operator splitting method in one dimension. We do this by considering the partial derivatives in one direction only until each sub-problem is solved each time , which is very possible with operator splitting method as discussed in the previous chapter. The 1-D solutions are compared with those in (Fodya, 2007) and (Grasselli & Hurd, 2007) only in terms of consistence with reference to the results in figure 4.1. The figure 4.2, shows that the expected option prices estimated through operator splitting method using the same model are in line with the expectation given by the volatility in figure 4.1. There is steady rise just like in prices of option prices.

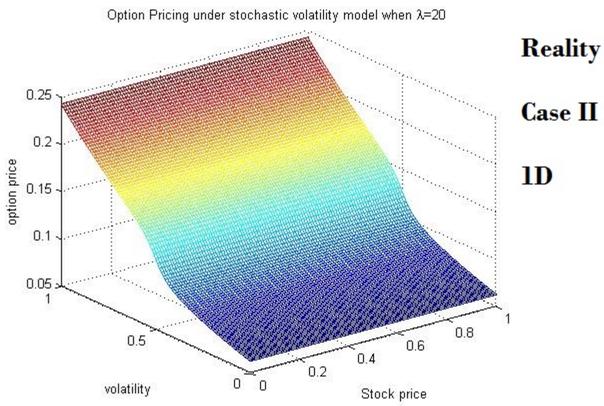

Figure 4.4: Option Prices in 1-D Through OSM when $\lambda = 20$

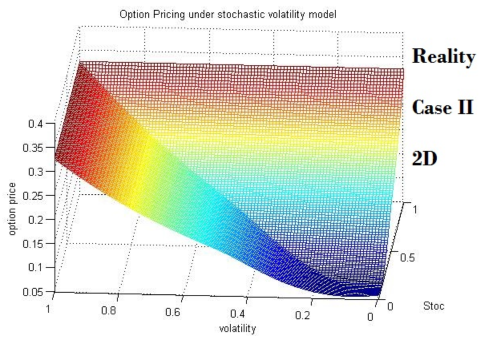

Figure 4.5: Option Prices in 2-D Through OSM when $\lambda = 20$

The result of finite difference method without splitting when $\lambda = 2$ from research done by Fodya (2007) are represented in figure A.1. The mesh plot (Figure 4.3) curves inwards which suggests existence of inconsistency of the solution of full method solution with the expectation set by the volatility in figure 4.1. This null hypothesis has been rejected in this section with strong evidence from the data at hand. Under the same condition that the jump parameter is freed to $\lambda = 2$, the solution (that is option price curve under stochastic volatility) via operator splitting technique in one dimension (Figure 4.2) flattens in the middle however permutations are changed in the method of solution (Appendix B). This firstly matches with the expectations set by Ikonen & Taivanen (2005) of truncation error order being invariant under different permutations. This also verifies the efficiency of the type of operator method adopted in this paper. The flattening (figure 4.2) suggests consistency in the operator method with reference to those in figure 4.1.

Since the results in (Fodya, 2007) represented in Figure A.1 are derived under the case of 2-D, it is imperative to evaluate the solution via operator splitting technique in 2-D for a fair comparison. This has been implemented by ensuring evaluation of partial derivatives in both axes at the same time (Appendix B) and represented by the results in the Figure 4.3. The shape is more flattened and more similar to 4.1. This suggests that there is no change in consistency under operator splitting method even if dimension changes.

Thus a more rigorous analysis of consistence of the operator splitting technique is needed. A sample of optional prices is collected when the underlying asset value is 0.5 maintaining the change in the volatility random variable in the same interval $[0, 1]$. A trend in the flow of the prices is statistically established via time series analysis techniques and the results in Figures A.8 and 4.6 are obtained.

The graph in Figure 4.6 suggests that there is an effect of standard volatility function on the price estimation given by operator splitting method. Since the step sizes in y are constant, it follows that the values of y cannot affect changes in price or volatility even if they are given as integers. A plot of y values represented as integers confirms this notion (Figure A.2). This justifies the use of the cross-correlation function (CCF) of two random processes $c = Y_t$ and $\sigma(y) = X_t$, which is the product moment correlation as a function of lag k , between the series' and it is defined as: $\rho_{xy} = \dfrac{\gamma_{xy}(s,t)}{\sqrt{\gamma_x(s,s)\gamma_y(t,t)}}$, which relates the kth cross-correlation coefficient, $\rho_{xy}(s,t)$, with the sample cross-covariance function ,$\gamma_{xy}(s,t)$, and the sample variances $\gamma_x(s,s)^{'}$s (Ross, 2009) to investigate the existence of effect of volatility when operator splitting method is applied to find the option prices in the model when the jump parameter is freed to 200.

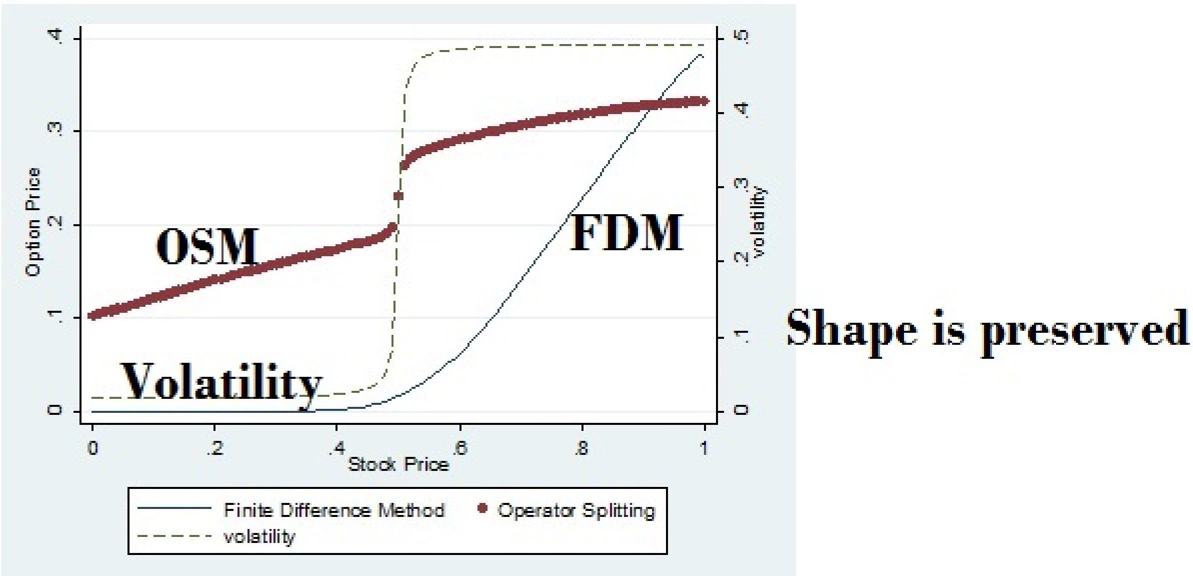

Figure 4.6: The effect of Volatility on Option Prices

The results in Figure A.8 do not need further analysis to investigate whether the correlations at each lag is zero or not, since at least 95% of the correlations are at least 0.5. This shows clearly now that the operator splitting method used in this research does not mask the impact of volatility on the optional price. In fact,

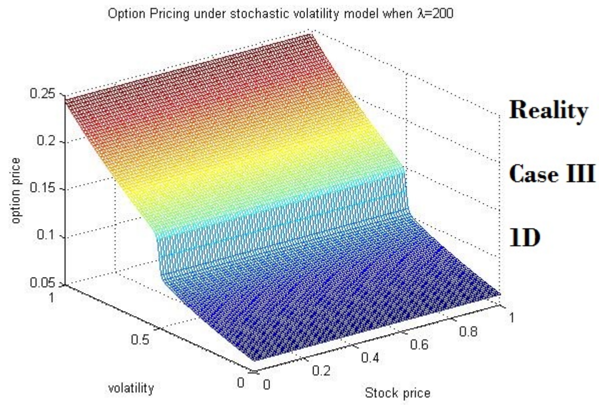

Figure 4.7: Option Prices in one Dimension Through OSM

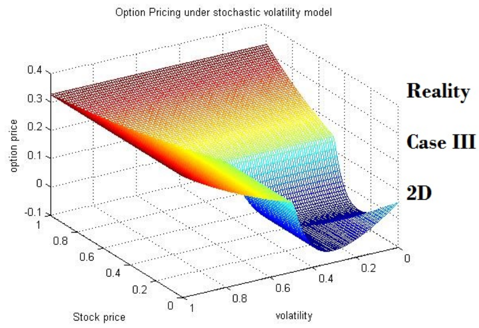

Figure 4.8: Option Prices in Two Dimensions Through OSM when $\lambda = 200$

there is a positive influence of volatility on price as expected (Grasselli & Hurd, 2007). These results are better than those under full method (Figure A.1).

We Sample from the data values in Figure A.3, which is obtained from (Fodya, 2007) and (Grasselli & Hurd, 2007). Performing similar analysis, the results in Figure A.5 are obtained. Cross-correlation function (Figure A.9) shows similar results that there is consistency in full method. Thus, the null assumption of inconsistency of the full method is rejected. At least 78% of correlations are at least 0.5. The only difference is that 78% is less than 95% and that at lag = 0, the correlation for the split method in this study is very high.

4.2 Computation Time Comparison

Table 4.1: Average Computational Times in Seconds

Number of Iterations	OSM Run	FDM Run
65	1.529	4044.3
81	1.57	5039.82
84	1.579	5226.48
92	1.604	5724.24
1000	19.055	62220

The computation time in seconds were collected from the operator splitting method and also sampled from full method in Fodya's run at different values of N (that is number of iterations) (Figure A.10). From this figure, it is very obvious that the running times are different and hence, no need to statistically verify the difference between the values. It can be seen (Figure A.10) that whenever $N \in [10, 300]$, computation time for the operator splitting method in this study is between 0.11 seconds and 31 seconds while in full method it is between 0.8 seconds and 7000 seconds. Similarly when $N = 1000$, computation time in operator splitting method is 972 seconds, but 52000 seconds in the full method (table 4.1). Thus computation times in operator splitting method are obviously lower than those in full method. It follows that a different way of comparing the data should be investigated. The advantage of full method is that the computational time is linear and a rerun of modeling in this study showed that the linear model is the most plausible model for their research (Fodya, 2007).

In figure 4.9, $b = (N2)^2$, $t2 =$ Computation Time in other research works (Full Method), $N2 = N$. These results suggest that a linear or a quadratic model is suitable. In fact, other types of models failed completely and we do not include them here.

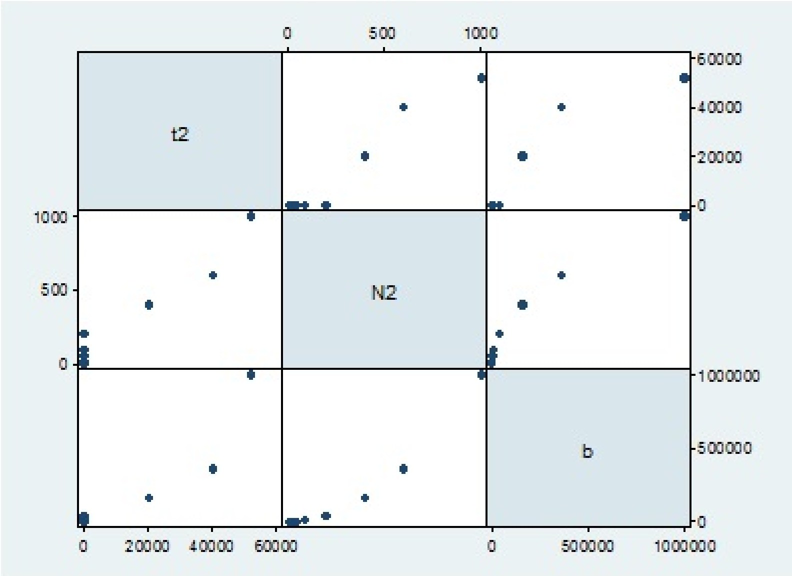

Figure 4.9: Modeling Computation Times in Full Method

However, the analysis of variance tables (Figures A.11 and A.12) show that both quadratic and linear models are suitable for modelling the relationship between computational times and N, since all the p-values are less than 0.05 for the observed F-statistic. However, a closer look at the p-values for the coefficients shows that p-values in quadratic model are bigger, F-statistic is smaller and adjusted R^2 is smaller. In particular, the p-value for $b = (N2)^2$ is 0.811 > 0.05, which means the leading coefficient of the assumed quadratic model (-0.005) is statistically not different from 0. This violates the definition of quadratic function. Therefore, the data at hand provides strong evidence that the linear model is suitable.

As for computational times in the operator splitting method used in this study (c=N^2), it is very obvious (Figure A.13) that the model relating this variable to the number of iterations is purely quadratic (R^2 = 99.55%) and not linear. To achieve a fair comparison, the models are compared on similar values. Thus times of computation for operator splitting method used in this study are lower than

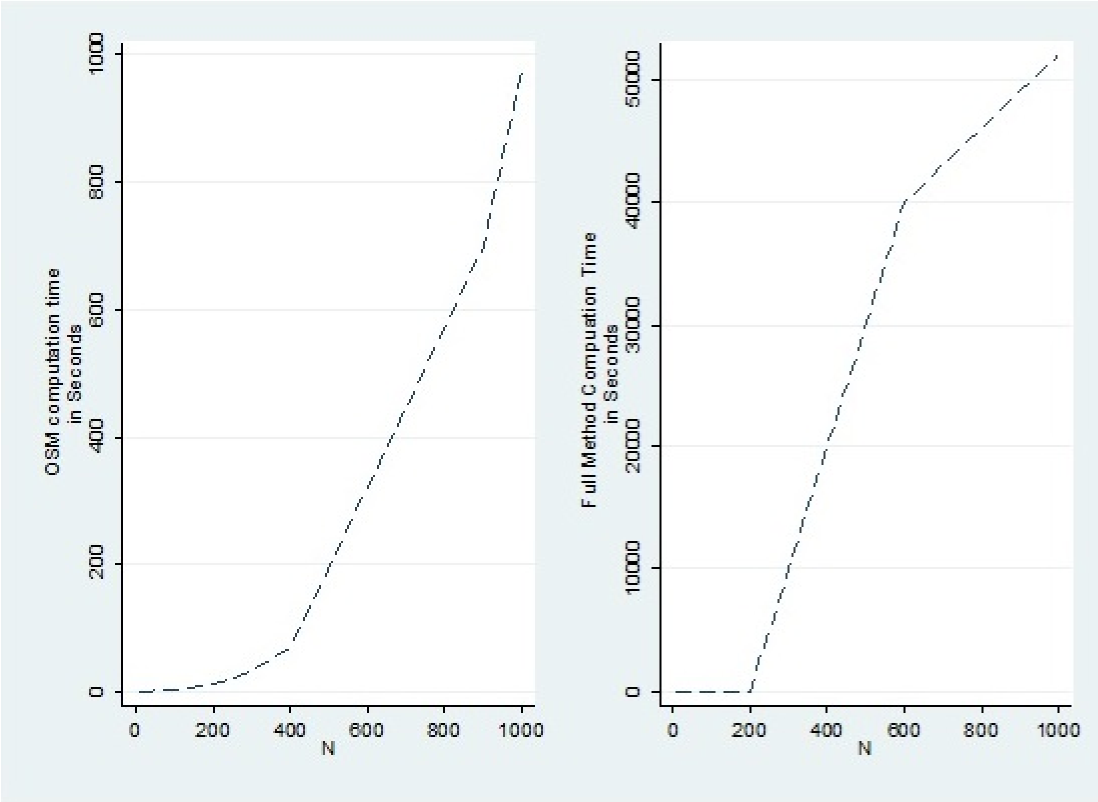

Figure 4.10: Computation Times Comparison

those in full method, but they are quadratic (Figure 4.10).

4.3 Predicting Actual Prices in the Market

A comparison is done here to find the closeness of price estimates in this paper to those actually demanded by buyers with reference to results given by Fodya (2007), and Grasseli & Hurd (2007). To do this data on observed market prices of Jet Fuel and options was collected from the Chicago Board of Options Exchange(CBOE) and Yahoo Finance (YF). The data spanned from the year 2000 to 2016. For some years it was collected in equally spaced time intervals. Data for the year 2004 was missing from the available sample. The data from Joshi (2008) book was used to verify accuracy. Focus was put on vanilla call options rather than put options.

The first exploration was made by plotting the observed prices of jet fuel options per litre against the observed stock prices per litre. The results showed mixed random patterns(Figure 4.11). A decision was made to separate the data

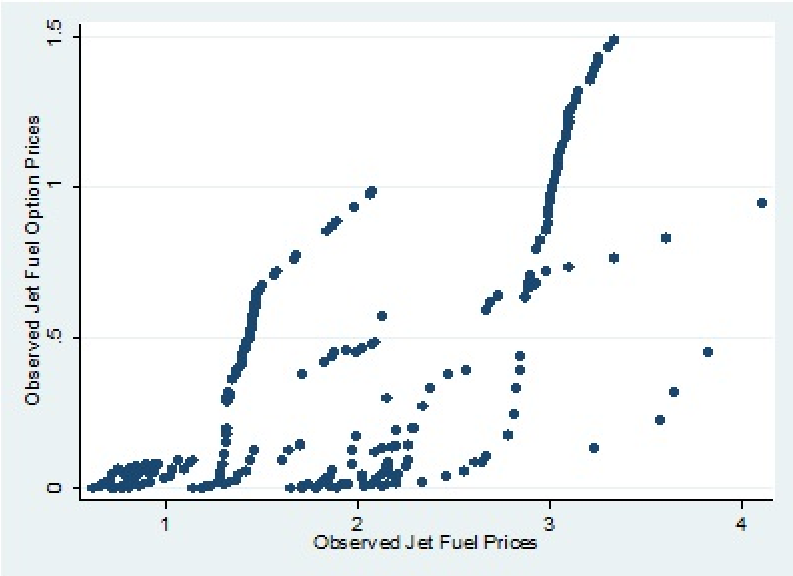

Figure 4.11: Jet Fuel Option Prices Observed in the Markets

values and study them in separate years. Non-random patterns were observed in this scenario (Figure A.4).

This is in line with the type of programming that was used to solve the problem (Appendix B) in which time was made constant to study the option prices in relationship with volatility random variable and the underlying prices. However, it has been noted that the prices in this relationship is periodic with period of one-year. The patterns are similar and repeated in each year. Focus was then put on the years 2011, 2012, 2015 and 2016, because there were a lot of data points in these years (Figure A.4).

These market values were compared with those computed via operator operator

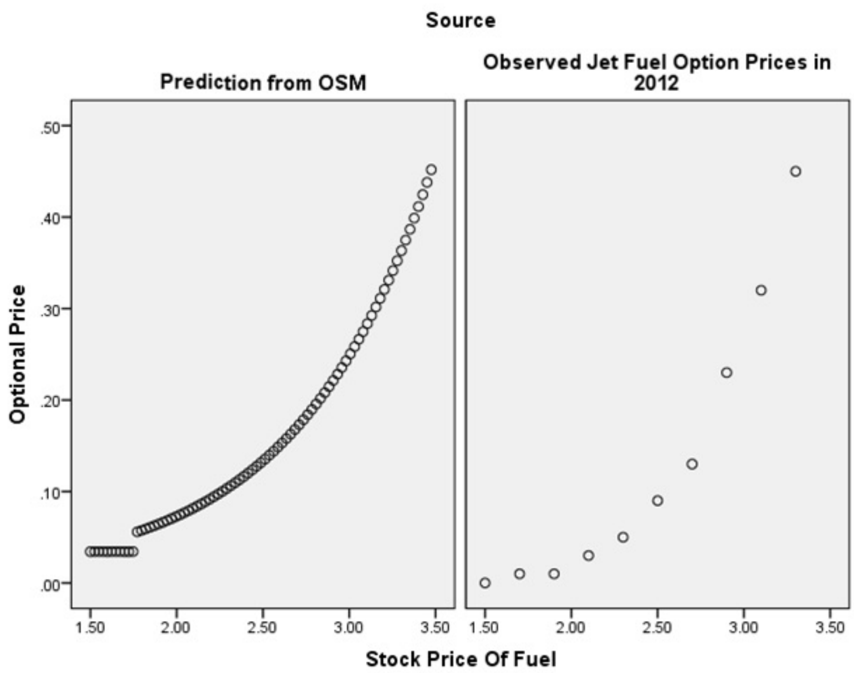

Figure 4.12: Investigating the Efficiency of OSM

Table 4.2: Numerical Consistency Check of Solution Via OSM

Year	Sample size=n	Number of Iterations=N	Relative error
2011	8	65	0.0145
2012	10	81	0.001101
2014	10	84	0.000923
2016	65	92	0.000457

splitting technique by measuring the degree of closeness on the predicted values got from the stochastic volatility model.

Analysis of Variance, Kruskal-Wallis test or T-test could not be used because the data sets whether predicted or observed were found not to be normally distributed (Figures A.14 and A.15) as predicted in the book of Ross (Ross, 2009), as well as they can not be assumed to be independent since they had been sampled at the same values of volatility for a fair comparison. For example, the model (Figure A.7) was used to predict market jet fuel option prices in the year 2012 (Figure A.6). It was noted that the value of $N = 81$ was needed to get the estimation of the observed price with a very low error of 0.001 (Figure 4.12) which is better than 0.1 when full method is used. That is to estimate these 10 points in Figure A.6 with this high degree of accuracy 81 points were taken in the model via operator splitting technique (Figure A.7). A number lower or higher than this lead to under or over estimation expanding the errors. Similar results were obtained in the trying to apply this model to predict option prices for the other years and the results are summarized (Table 4.2). Thus, all these computations to predict optional prices took less than 0.966 second to run completely, because $N < 100$ (Table 4.2 and Figure 4.10)

Chapter 5

CONCLUSION AND FURTHER WORK

5.1 Conclusion

5.1.1 Introduction

This paper has considered one way of improving the works of Fodya (2007) and Betteridge (2005) by using operator splitting method in order to price options using the stochastic differential equations. We considered the problem of solving the stochastic volatility model instead of Black-Scholes and Binomial models, because they are constructed under the risk neutrality assumption and hence, they do not give computationally realistic prices. The the stochastic volatility model used by Fodya (2007) and Betteridge (2005) has stochastic volatility assumption included through utility function.

There is no closed solution to the stochastic volatility model of option pricing. Most researchers use finite difference and Monte Carlo Methods to find the approximate solutions of the model. It has been noted that the early users of finite difference method quoted in this thesis used the Feynman-Kac formula to convert a stochastic differential equation into a partial differential equation. This is sufficient to solve the stochastic volatility equation up to two dimensions. In higher dimensions the method faces the curse of dimensionality discussed in introductory chapters of this paper. The immediate answer to the problem has been to use the Monte-Carlo method for higher dimensions. It has been noted that Monte Carlo methods face convergence problems in that they take long time to converge. This problem has usually been solved by either developing a new form of Monte Carlo method or by modifying assumptions in the stochastic volatility model and then use Monte Carlo or finite difference methods. This has brought in many problems like increasing relative error when strike price of option increases.

5.1.2 Achieving First and Second Objective

In this study, an operator splitting technique with five distinct split sub-problems is adopted. The sixth sub-problem is a weighted form of one of the sub-problems. This is a principle in Strang's recombination scheme. The first four sub-problems do not have closed forms. They are solved using Runge-Kutta method. The partial derivatives are expressed in finite difference form and the Runge-Kutta method is applied on these forms to estimate solutions to the split sub-problems. This has been shown to be an improvement to the finite difference method. The last sub-problem is non-linear and has a closed form under the Charpit's method.

We considered doing this in order to use the most appropriate technique in each sub-problems and hence make the operator splitting technique efficient. This allowed us to construct computationally Runge-Kutta allows for the whole program to be cheaper in terms of computer processing unit run time (Langtangen, 2012).

This achieves the second objective and partly the last objective of this study, because it allowed for use of simple Mathematics Laboratory commands instead of hard coded solvers for differential equations.

Theoretically, it has been shown that the operator splitting method used in this paper is consistent, if the all the parametric coefficients of the stochastic differential equation are not zero. The schemes are also found to be stable, if the Courant-Fredrick-Levy numbers computed through Von-Neuman stability analysis should not be zero. To achieve empirical stability and consistent we used all the values given in the literature (Fodya, 2007; Betteridge, 2005; Grasselli & Hurd, 2007).

5.1.3 Achieving the Third Objective

Thus, the operator splitting technique used in this research has been found to be both theoretically and experimentally stable, consistent and convergent. This is so because of the use of the Strang's method that has second order accuracy of truncation error. The Runge-Kutta method also allowed for the finite differences not to distort the physical meaning of the explicitly expressed partial derivatives in the stochastic volatility model. This explains why the shapes remained unchanged under different permutations. This means the operator splitting method used in this study gives computationally more realistic than the finite difference method used without splitting.

In addition, we have found that the truncation error is not increasing because of splitting, and that the method is fast. For example, for 1000 iterations, the operator splitting method runs in a time that is less than 16 minutes, but the finite difference method runs in a time that is more than 14 hours. This happens when there is a large jump in volatility. In this case we see that operator splitting technique used in this study is 52 times cheaper in demand for time resources.

It has been seen that for smaller number iterations the operator splitting technique computation times are comparable with finite difference method, but for larger number iterations the computation times are still lower in operator splitting technique, but expand linearly in finite difference method. The times in operator splitting rise very slowly. Apart from very small number of iterations, there clearly exists a number of iterations for which the operator splitting technique used in this thesis and the finite difference method have the same high computation time. However, the study has clearly shown that such a number is infinitely large for practical purposes. It has been shown in this study that efficiency can be easily improved, by simply choosing the most appropriate method for a sub-problem.

It has also been seen that the operator does not mask the effect of volatility much better than the finite difference method. This is because of the fact that the Strang's method a midst Runge-Kutta method and also the fact that very appropriate finite different schemes were selected for each sub-problem. If follows that in the light of the standard volatility function the operator splitting technique provides numerical solution that are very close to the unknown solution of the stochastic volatility model. That is the operator splitting method helps one estimate realistic option prices using the realistic models. This fact has also been evidenced from the fact that the estimates of prices obtained in this study are very close to those accepted on the market collected by Chicago Board of Option Exchange. The relative errors are also not increasing with increase in number of iterations or strike price. Thus, the method is computationally realistic.

5.1.4 Summary

Thus, we intended to explore the efficiency of operator splitting technique. To do this, we solved the stochastic volatility model (3.1) using operator splitting technique. We have achieved this specific objective by applying finite difference representation of partial derivatives, Charpit's method, Runge-Kutta method, and Strang's recombination Scheme to the five split sub-problems from the model. We then constructed an efficient Matrix Laboratory program to run the numerical method of the scheme. To achieve the third objective, we compared the results are compared with those of Fodya (2007), Betteridge (2005), and Grasselli & Hurd (2007). We have seen that the operator splitting method is more computationally realistic and cheaper in terms of demand for time resources. In other words, the method is more efficient.

5.2 Further Work

This paper has solved the stochastic volatility models up to two dimensions. We have shown that the method is efficient in that it gives realistic results within polynomial time complexities. Another researcher might think of tackling a higher dimension problem. In addition, we considered using Runge-Kutta together with very appropriate finite difference methods. One could think of excluding the

Runge-Kutta method, and then use recent hybrid Monte Carlo methods (Faou, 2011) for each sub-problem in three dimensions. Since we are studying stochastic volatility model for option pricing by splitting the operators randomly, this study clearly shows that presentation of stochastic volatility model for option prices as a set of algorithms rather that one unique formula could be an interesting problem that could be subject of another study. One can also analyse the operator splitting much deeper in stochastic processes.

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Appendix A

Data Used in the Study

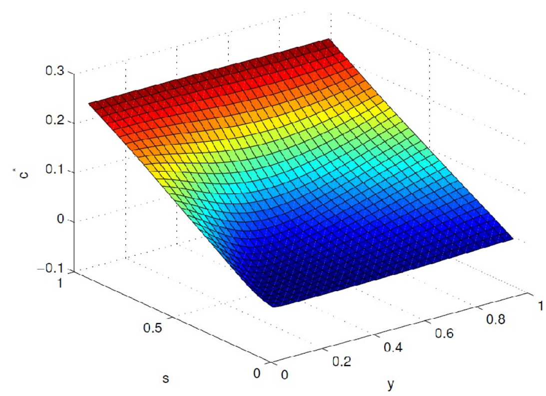

Figure A.1: Full Method Option Prices in 2-D $\lambda = 2$

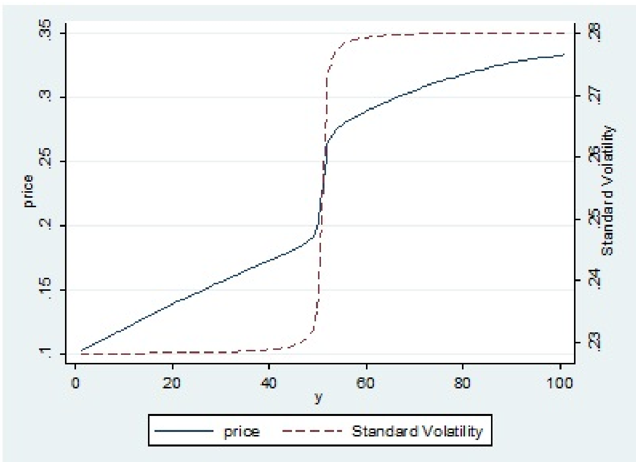

Figure A.2: The effect of volatility on OSM Solution

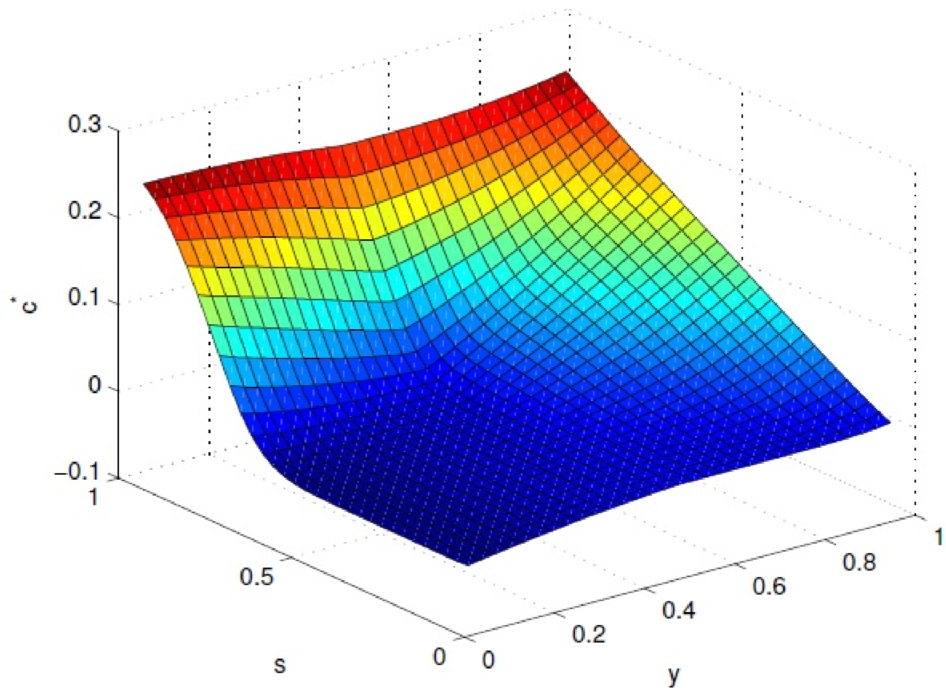

Figure A.3: Full Method Optional Prices in 2-D $\lambda = 200$

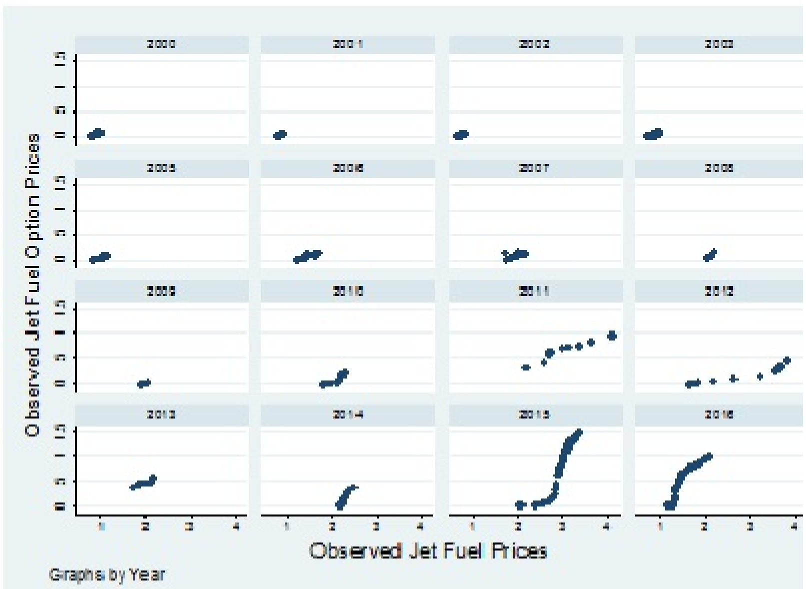

Figure A.4: Jet Fuel Option Prices

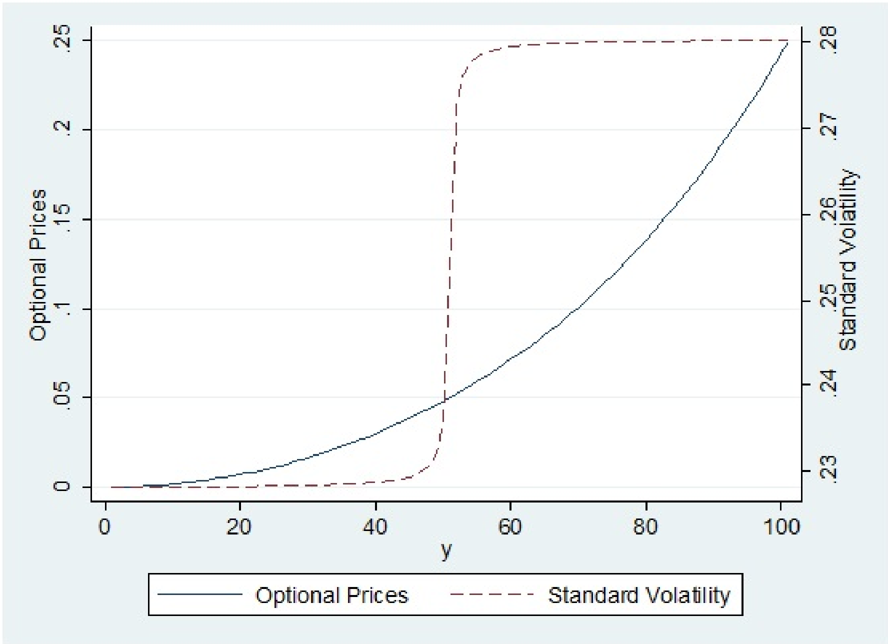

Figure A.5: Impact of Volatility on FDM Option Prices

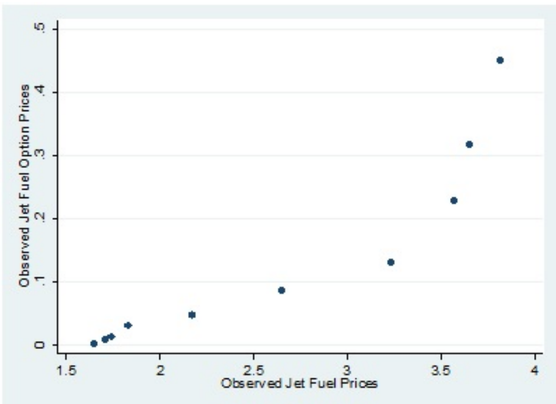

Figure A.6: 2012 Jet Fuel Optional Prices

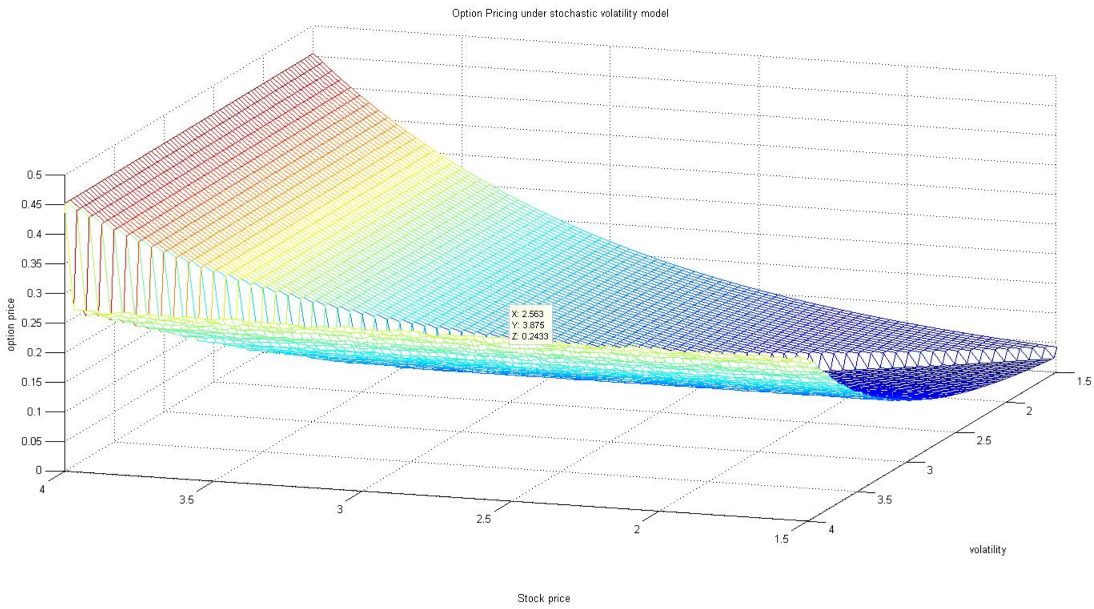

Figure A.7: Predicting 2012 Jet Fuel Optional Prices in 2-D $\lambda = 200$

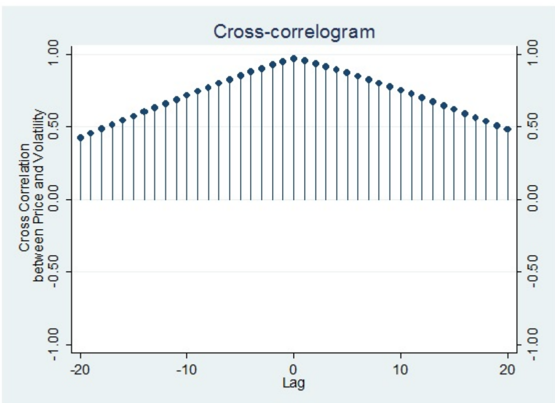

Figure A.8: Verifying Volatility Impact on OSM Optional Prices

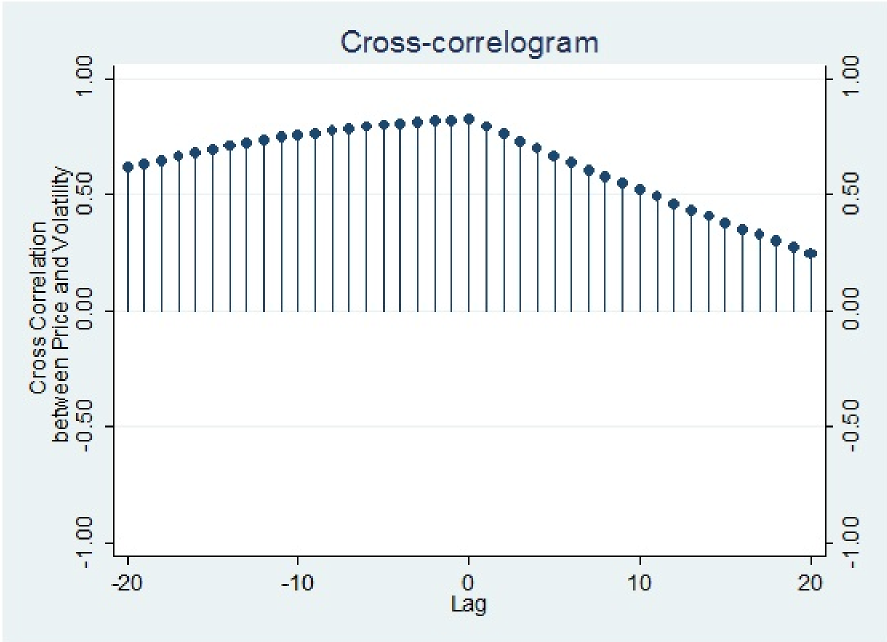

Figure A.9: Correlation between FDM Option Prices and Volatility

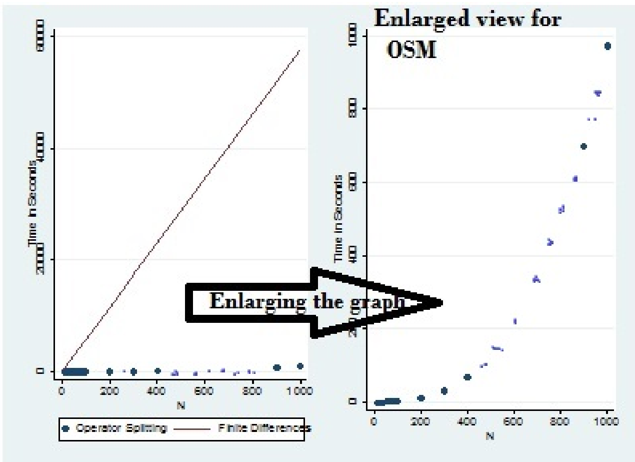

Figure A.10: Computation Time for OSM and Full Method

```
      Source |       SS       df       MS              Number of obs =       9
-------------+------------------------------           F(  1,     7) =  128.37
       Model |  3.1389e+09     1  3.1389e+09           Prob > F      =  0.0000
    Residual |   171159802     7  24451400.2           R-squared     =  0.9483
-------------+------------------------------           Adj R-squared =  0.9409
       Total |  3.3101e+09     8   413762534           Root MSE      =  4944.8

------------------------------------------------------------------------------
          t2 |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
          N2 |    57.6682   5.089753    11.33   0.000     45.63284    69.70355
       _cons |  -2779.415   2126.579    -1.31   0.233    -7807.975    2249.145
------------------------------------------------------------------------------
N2=N
t2=computation time
```

Figure A.11: Assuming Linear Model

```
      Source |       SS       df       MS              Number of obs =       9
-------------+------------------------------           F(  2,     6) =   55.62
       Model |  3.1407e+09     2  1.5704e+09           Prob > F      =  0.0001
    Residual |   169392346     6  28232057.7           R-squared     =  0.9488
-------------+------------------------------           Adj R-squared =  0.9318
       Total |  3.3101e+09     8   413762534           Root MSE      =  5313.4

------------------------------------------------------------------------------
          t2 |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
          N2 |   62.22067    18.9989     3.27   0.017     15.73204    108.7093
           b |  -.0049159   .0196472    -0.25   0.811     -.052991    .0431592
       _cons |  -3123.097    2666.14    -1.17   0.286    -9646.907    3400.712
------------------------------------------------------------------------------
```

Figure A.12: Assuming Quadratic Model

Source	SS	df	MS
Model	1398614.52	2	699307.261
Residual	6244.81599	88	70.9638181
Total	1404859.34	90	15609.5482

Number of obs	=	91
F(2, 88)	=	9854.42
Prob > F	=	0.0000
R-squared	=	0.9956
Adj R-squared	=	0.9955
Root MSE	=	8.424

time	Coef.	Std. Err.	t	P>|t|	[95% Conf.	Interval]
c	.0012299	.0000176	69.71	0.000	.0011948	.0012649
N	-.3061102	.0155042	-19.74	0.000	-.3369215	-.2752989
_cons	13.26847	1.455396	9.12	0.000	10.37617	16.16076

Figure A.13: Modelling Computational Times in OSM

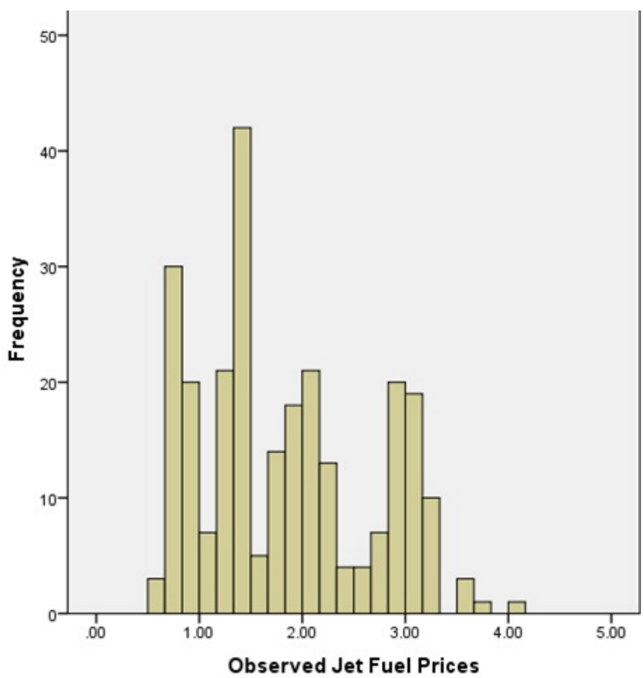

Figure A.14: Checking Normality Observed Jet Fuel Prices

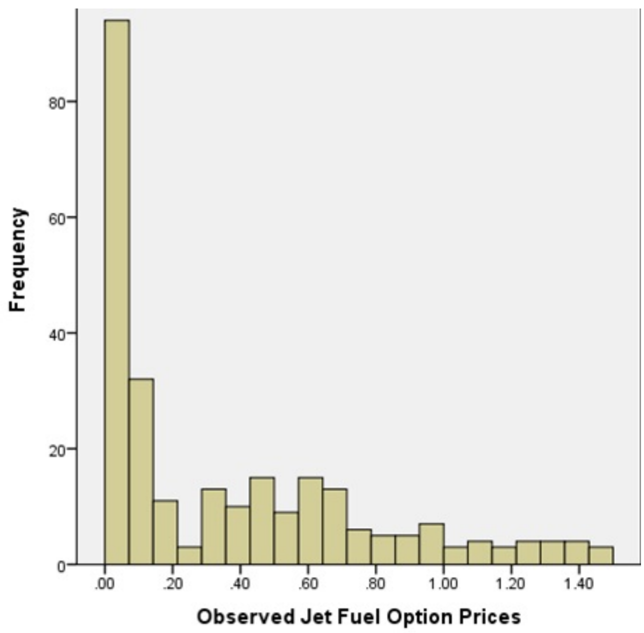

Figure A.15: Checking Normality of Jet Fuel Option Prices

Appendix B

Matrix Laboratory Program for OSM

B.1 2-D Programming

```
function fodya6

h=0.1;

t1=1;

t2=0.001;

x1=0:h^2:t1;

n=x1';

m=length(n);

A=zeros(m);

B=zeros(m);

for i=1:m

    A(:,i)= firstsol(x1(i),x1,t2);

    u=A(:,i);

    u1=u;

    dudt=-5*0.5.*(x1(i))^2*(vola(x1(i)))...

    ^2.*exercise3(m,u);  %evaluating the

    finite differences
```

```
of 0.5s^2 sigma^2 u_{ss}

u=dudt;

s=0.1;

a11= s.*exercise3(m,u);

u=u+0.5.*a11; % The process of

evaluating the solution of the

first split subproblem has

started here

a21=s.* exercise3(m,u);

u=u1+ 0.5.* a21;

a31=h.*exercise3(m,u);

dudt=u1+(1/6).*a11+(1/3).*a21...

    +(1/6).*a31;

B(:,i)=dudt;

u=B(:,i)';

b=0.3;

dudt=-0.5*b^2.*ex3(m,u); %evaluating

the finite differences of 0.5s^2 u_{yy}

u=u1'+dudt;

% Now we are combining

the solutions with those

of the second split subproblem

 via strang

b11=s.*exercise3(m,u);
```

```
u=u+0.5.*b11;

b21=s.*exercise3(m,u);

u=u1'+0.5.*b21;

b31=h.*exercise3(m,u);

dudt=u1' + (1/6).*b11+(1/3).*b21 + (1/6).*b31;

B(i,:)=dudt;

 % Now we are combining

  the solutions with those

   of the third split subproblem

    via strang

u=B(i,:)';

dudt= ex3(m,u);

u=dudt';

b=0.3;

a=0.5;

RHO=0.5;

dudt=-a*b*RHO*0.25*2*x1(i).*ex3(m,u);

B(i,:)=dudt;

u=u1+B(i,:)';

dudt=ex3(m,u);

 u=dudt';

dudt=ex3(m,u);

c11=s.*dudt;

u=u+0.5.*c11;
```

```
  u=u1+u';

  dudt=ex3(m,u);

   u=dudt';

  dudt=ex3(m,u);

  c21=s.*dudt;

  u=u+0.5.*c21;

  u=u1+u';

  dudt=ex3(m,u);

   u=dudt';

  dudt=ex3(m,u);

  c31=h.*dudt;

  dudt=u1' + (1/6).*c11+(1/3).*c21 + (1/6).*c31;

  B(i,:)=dudt;

%   Now combining the

 solution to those of the fourth

% split subproblem(20-07-2016)

 at 12:15 PM wednessday

  b=0.3;

  a=0.5;

  RHO=0.5;

  u=B(i,:);

  drift=0.04;

  r=0.02;

  dudt=-a+RHO*(drift-r)*b.*ex3(m,u);
```

```
u=dudt;

s=0.1;

d11= s.* ex3(m,u);

u=u+0.5.*d11; %

The process of evaluating the

solution of the fourth split s

ubproblem has started here

  d21=s.* ex3(m,u);

u=u1'+ 0.5.* d21;

d31=s.* ex3(m,u);

dudt=u1'+(1/6).*d11+(1/3).*d21...

      +(1/6).*d31;

 B(i,:)=dudt;

 u=B(i,:);%Charpits method starts here

 u3=u;

 CONST=u-(drift-r).^2/((vola(x1(i)).^2)*2).*x1(i).*ones(1,m);

 u=u.*x1(i)+CONST;

 B(i,:)=u;

 u=B(i,:)';

 k=(drift-r).^2/((vola(x1(i)).^2)*2);

 u=a.*2.*(1/m).*u+u3';

 u=sqrt(a/k).*2.*(1/m).*u'+u3;

 B(i,:)=u;

end
```

```
% % for i=1:floor(0.2*m*2)+1;
% %     B(m-i+1,1:m)=u1;
%This loop
is deleting the values
 that fall out of range due
  to differencing and consequent
   padding of zeroes
% % end
mesh(x1,x1,B')
xlabel('Stock price ');   ylabel('volatility');   zlabel('option price');
title('Option Pricing under stochastic volatility model');

function y=ex3(m,w)
a=1;
b=m;
for i=1:m
    k=a+(b-a)*rand(1,m);
    t=floor(k);
    z=(w(i)-w(t)).*(1/.2);
end
y=z;


function z=exercise3(m,w)
a=1;
b=m;
```

```
for i=1:m

    k=a+(b-a)*rand(m,1);

    t=floor(k);

    z=(w(i)-w(t)).*(1/0.2);

end

function u=firstsol(x,s1,t)

u= 0.002*erf(s1*t+x)+0.01*cosh(x-s1*t);

clc;
```

B.2 1-D Programming

```
function fodya_SecondPermutation

h=0.1;

t1=1;

t2=0.00000001;

x1=0:h^2:t1;

n=x1';

m=length(n);

A=zeros(m);

B=zeros(m);

i=0;

while i<m

    i=i+1;

    A(:,i)= firstsol(x1(i),x1,t2);

    u=A(:,i);
```

```
    u=AL(u',i);


  % Now we are
  combining the solutions
   with those of the second
   split subproblem via strang
    u=BL(u,i);


   % Now we are combining
   the solutions with those
    of the third split
    subproblem via strang
   u=CL(u,i);


%   Now combining the
solution to those of the fourth
% split subproblem(20-07-2016)
 at 12:15 PM wednessday
  u=DL(u,i);


   B(i,:)=u;
end
 B;
```

```
mesh(x1,x1,B)

xlabel('Stock price ');   ylabel('volatility');   zlabel('option price');

title('Option Pricing under stochastic volatility model when \lambda=2');


filename = 'pat.xlsx';

xlswrite(filename,B,'A1:CW101')




function s=vola(y)

lambda=200;

s= (0.25+(1/(2*pi))*atand( lambda *(y-0.5))+15)*.5/30;

function u=CL(u,i)

% input vector should be row

h=0.1;

s=h;

t1=1;

x1=0:h^2:t1;

B=[];

m=length(u);

B(i,:)=u;

u=B(i,:)';

  dudt= ex3(m,u);

  u1=u;

  u=dudt';
```

```
b=0.3;
a=0.5;
RHO=0.5;
dudt=a*b*RHO*0.25*2*x1(i).*ex3(m,u);
B(i,:)=dudt;
u=u1+B(i,:)';
dudt=ex3(m,u);
 u=dudt';
dudt=ex3(m,u);
c11=s.*dudt;
u=u+0.5.*c11;
u=u1+u';
dudt=ex3(m,u);
 u=dudt';
dudt=ex3(m,u);
c21=s.*dudt;
u=u+0.5.*c21;
u=u1+u';
dudt=ex3(m,u);
 u=dudt';
dudt=ex3(m,u);
c31=h.*dudt;
dudt=u1' + (1/6).*c11+(1/3).*c21 + (1/6).*c31;
B(i,:)=dudt;
```

```
  u=B(i,:);


function U=CL1(M,u)
%for heat eqution
N= floor(M);
T=1;
dx=1/M;
dt=T/N;
lambda=dt/(dx)^2;
U=zeros(N+1,M+1);
A=speye(M-1)+lambda *...
    spdiags([-ones(M-1,1),2*ones(M-1,1), -ones(M-1,1)], [-1,0,1],M-1,M-1);
 for j=1:M+1
     U(1,j)=u(1,j);
 end
 for n=1:N
     U(n+1,1)=0;
     x=A\U(n,2:M)';
     U(n+1,2:M)=x';
     U(n+1,M+1)=0;
 end
 U';
 mesh(0:dx:1,0:dt:1,U')
```

```
function u=BL(u,i)
% Input vector should be row
m=length(u);
s=0.1;
h=s;
u1=u';
B=zeros(m);
b11=s.*exercise3(m,u);
  u=u+0.5.*b11;
  b21=s.*exercise3(m,u);
  u=u1'+0.5.*b21;
  b31=h.*exercise3(m,u);
  dudt=u1' + (1/6).*b11+(1/3).*b21 + (1/6).*b31;
  B(i,:)=dudt;
  u=B(i,:);


function z=DL(u,i)
h=0.1;
t1=1;
x1=0:h^2:t1;
m=length(u);
B=zeros(m);
B(i,:)=u;
```

```
u1=u';

b=0.3;

  a=0.5;

  RHO=0.5;

  u=B(i,:);

  drift=0.04;

  r=0.02;

  dudt=-a-RHO*(drift-r)*b.*ex3(m,u);

  u=dudt;

  s=0.1;

  d11= s.* ex3(m,u);

  u=u+0.5.*d11;

  % The process of

  evaluating the

  solution of the fourth

  split subproblem has

  started here

    d21=s.* ex3(m,u);

  u=u1'+ 0.5.* d21;

  d31=s.* ex3(m,u);

  dudt=u1'+(1/6).*d11+(1/3).*d21...

        +(1/6).*d31;

    u=B(i,:);%Charpits method starts here

   u3=u;
```

```
   CONST=u-(drift-r).^2/((vola(x1(i)).^2)*2).*x1(i).*ones(1,m);
   u=u.*x1(i)+CONST;
   B(i,:)=u;
   u=B(i,:)';
   k=(drift-r).^2/((vola(x1(i)).^2)*2);
   u=a.*2.*(1/m).*u+u3';
   u=sqrt(a/k).*2.*(1/m).*u'+u3;
   B(i,:)=u;
   z=B(i,:);


function z=AL(u,i)
%input vector
h=0.1;
t1=1;
x1=0:h^2:t1;
A=[];
B=[];
    A(:,i)= u;
    u=A(:,i);
    u1=u;
    m=length(u);
    dudt=-5*4.*(x1(i))^2*(vola(x1(i)))^2.*exercise3(m,u);  %evaluating the
    finite differences
```

```
of 0.5s^2 sigma^2 u_{ss}
u=dudt;
s=0.1;
a11= s.*exercise3(m,u);
u=u+0.5.*a11;
 % The process of
  evaluating the solution
   of the first split
   subproblem
   has started here
a21=s.* exercise3(m,u);
u=u1+ 0.5.* a21;
a31=h.*exercise3(m,u);
dudt=u1+(1/6).*a11+(1/3).*a21...
    +(1/6).*a31;
B(:,i)=dudt;
u=B(:,i)';
b=0.3;
dudt=-0.5*b^2.*ex3(m,u);
 %evaluating the finite
 differences of 0.5s^2 u_{yy}
z=u1'+dudt;


function U=AL1(M)
```

```
%for heat eqution

N= floor(M);

T=1;

dx=1/M;

x1=0:dx:1;

dt=T/N;

lambda=dt/(dx)^2;

U=zeros(N+1,M+1);

A=speye(M-1)+lambda *...

    spdiags([-ones(M-1,1),2*ones(M-1,1), -ones(M-1,1)], [-1,0,1],M-1,M-1);

 for j=1:M+1

     U(1,j)=phi((j-1)*dx);

 end

 for n=1:N

     U(n+1,1)=0;

     x=A\U(n,2:M)';

     U(n+1,2:M)=5*4.*(x1(n))^2*(vola(x1(n)))^2.*x';

     U(n+1,M+1)=0;

 end

 U;


 mesh(x1,x1,U)
```